

4.5 Probability generating functions

Name: _____ Class: _____ Date: _____

Total: 14 marks

Objective

Build the skills to answer exam questions on **probability generating functions** 概率母函数—building $G(t)$, getting the mean and variance from its derivatives, and combining independent variables.

You must be able to:

- write $G(t) = E(t^X) = \sum P(X = x)t^x$
- use $E(X) = G'(1)$ and $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$
- multiply PGFs for a sum of independent variables

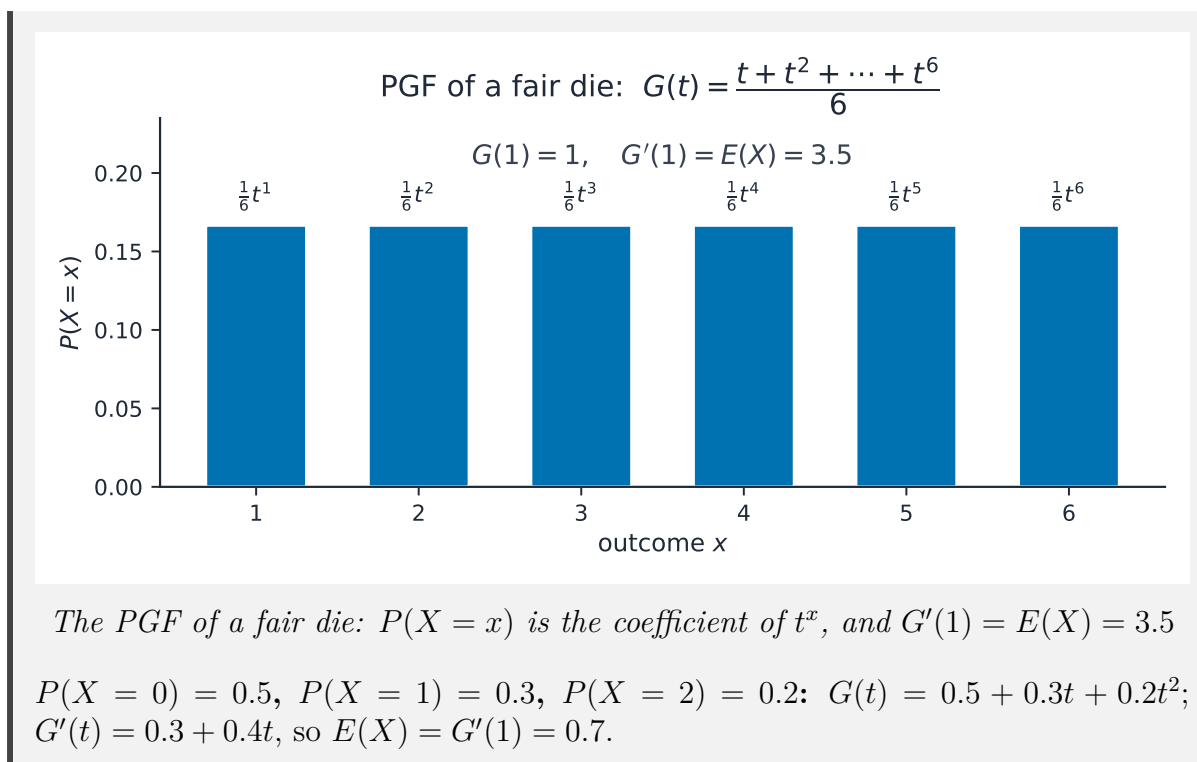
1 Worked examples

■ Mean from the PGF



A PGF packs a whole discrete distribution into one function $G(t)$

Image: source



2 Practice

2.1 Write down $G(t)$ for a variable with $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$. [2]

2.2 State the formula for $E(X)$ in terms of G . [1]

3 Exam-style questions

3.1 For any PGF, $G(1)$ equals: [1]

- A 0
- B $E(X)$
- C 1
- D $\text{Var}(X)$

3.2 A variable has $G(t) = \frac{1}{4}(1 + t)^2$.

(a) Find the probability distribution of X . [3]

(b) Find $E(X)$ and $\text{Var}(X)$. [4]

3.3 X and Y are independent with PGFs $G_X(t) = \frac{1}{2}(1 + t)$ and $G_Y(t) = \frac{1}{3}(1 + t + t^2)$. Find the PGF of $X + Y$ and hence $P(X + Y = 0)$. [3]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **4.5 Probability generating functions** past-paper sheet in the Library, or try these in **Practice mode**:

- 9231/41 N25 —Q7 (PGF)
- 9231/42 N25 —Q6 (PGF mean/variance)
- 9231/44 N25 —Q4 (generating function)

Solutions

2.1 $G(t) = \frac{1}{3}(t + t^2 + t^3).$

2.2 $E(X) = G'(1).$

3.1 C. $G(1) = \sum P(X = x) = 1.$

3.2 (a) $G(t) = \frac{1}{4}(1 + 2t + t^2) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$, so $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$.

(b) $G'(t) = \frac{1}{2} + \frac{1}{2}t \Rightarrow E(X) = G'(1) = 1$; $G''(t) = \frac{1}{2} \Rightarrow \text{Var}(X) = \frac{1}{2} + 1 - 1^2 = \frac{1}{2}.$

3.3 $G_{X+Y}(t) = G_X(t)G_Y(t) = \frac{1}{2}(1+t) \cdot \frac{1}{3}(1+t+t^2) = \frac{1}{6}(1+t)(1+t+t^2)$; $P(X+Y=0)$ is the constant term $= \frac{1}{6}.$