

4.1 Continuous random variables

Name: _____ Class: _____ Date: _____ Total: 14 marks

KEY WORDS probability density function 概率密度函数
• cumulative distribution function 累积分布函数 • median 中位数 • percentiles 百分位数
• continuous random variables 连续型随机变量

Objective

Build the skills to answer exam questions on **continuous random variables** 连续型随机变量 — the **probability density function** 概率密度函数, $E(g(X))$, and the **cumulative distribution function** 累积分布函数.

You must be able to:

- find a constant / probability from a pdf (possibly piecewise)
- find $E(X)$, $\text{Var}(X)$ and $E(g(X))$ by integration
- find $F(x)$ and use it for the median and percentiles

1 Worked examples

■ pdf, cdf and median

area to the left = 0.5

$y = f(x)$

median $m = \sqrt{2}$

median m solves $F(m) = \int_{-\infty}^m f(x) dx = \frac{1}{2}$

The median is where the area under $f(x)$ to its left is 0.5

$F(x) = 1$ at the top

$F(m) = 0.5$

$F(x)$

x median m

$F(x) = P(X \leq x) = \int_{-\infty}^x f \cdot 0 \leq F \leq 1 \cdot \text{median: } F(m) = \frac{1}{2}$

$F(x)$ rises from 0 to 1; the median is where $F(x) = 0.5$

$f(x) = \frac{1}{2}x$ on $[0, 2]$; $F(x) = \frac{1}{4}x^2$; median $\frac{1}{4}m^2 = 0.5 \Rightarrow m = \sqrt{2}$.

2 Practice

2.1 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $E(X)$. [2]

2.2 Write the relationship between $f(x)$ and $F(x)$. [1]

3 Exam-style questions

3.1 The median of a continuous variable solves: [1]

- A $f(m) = 0.5$
- B $F(m) = 0.5$
- C $E(X) = m$
- D $F(m) = 1$

3.2 A continuous variable has $f(x) = kx^2$ for $0 \leq x \leq 3$ (and 0 elsewhere).

(a) Find k . [3]

(b) Find $E(X)$. [3]

3.3 A variable has cdf $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$.

(a) Find the pdf $f(x)$. [2]

(b) Find the median. [2]

Solutions

$$\mathbf{2.1} \quad E(X) = \int_0^2 x \cdot \frac{1}{2}x \, dx = \left[\frac{x^3}{6} \right]_0^2 = \frac{4}{3}.$$

$$\mathbf{2.2} \quad f(x) = F'(x).$$

3.1 B. The median solves $F(m) = 0.5$.

$$\mathbf{3.2} \quad (\text{a}) \quad \int_0^3 kx^2 \, dx = \left[\frac{kx^3}{3} \right]_0^3 = 9k = 1 \Rightarrow k = \frac{1}{9}.$$

$$(\text{b}) \quad E(X) = \int_0^3 x \cdot \frac{1}{9}x^2 \, dx = \frac{1}{9} \left[\frac{x^4}{4} \right]_0^3 = \frac{1}{9} \cdot \frac{81}{4} = \frac{9}{4}.$$

$$\mathbf{3.3} \quad (\text{a}) \quad f(x) = F'(x) = \frac{x}{8} \text{ for } 0 \leq x \leq 4.$$

$$(\text{b}) \quad \frac{m^2}{16} = 0.5 \Rightarrow m^2 = 8 \Rightarrow m = 2\sqrt{2} = 2.83.$$