

2.6 Differential equations

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS general solution 通解 • integrating factor 积分因子 • auxiliary equation 辅助方程
• complementary function 余函数 • particular integral 特积分

Objective

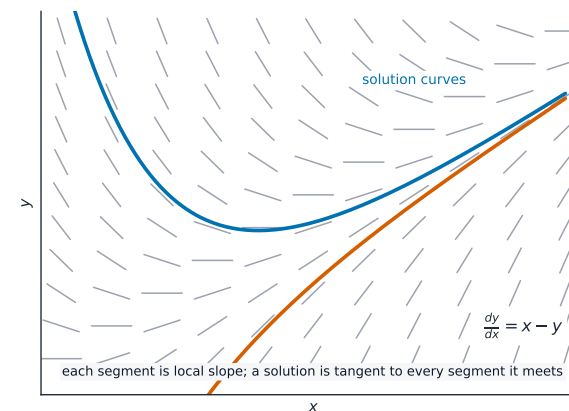
Build the skills to answer exam questions on **differential equations** 微分方程 — the **integrating factor** 积分因子 for first-order linear equations, and the **complementary function** 余函数 plus **particular integral** 特积分 for second-order constant-coefficient equations.

You must be able to:

- solve $\frac{dy}{dx} + P(x)y = Q(x)$ with $\mu = e^{\int P dx}$
- solve $ay'' + by' + cy = 0$ via the auxiliary equation (the CF)
- find a particular integral and the general solution

1 Worked examples

- Integrating factor & CF+PI



A solution curve stays tangent to the slope field everywhere

$$\frac{dy}{dx} + 2y = e^x: \mu = e^{2x}; \frac{d}{dx}(ye^{2x}) = e^{3x}, \text{ so } y = \frac{1}{3}e^x + Ce^{-2x}.$$

Second order: $y'' - 5y' + 6y = 0 \rightarrow$ auxiliary $m^2 - 5m + 6 = 0 \rightarrow y = Ae^{2x} + Be^{3x}$.

2 Practice

2.1 Write down the integrating factor for $\frac{dy}{dx} + 3y = x$. [1]

2.2 Write the auxiliary equation of $y'' - y' - 6y = 0$. [1]

3 Exam-style questions

3.1 The complementary function of $y'' + 4y = 0$ is: [1]

- A $Ae^{2x} + Be^{-2x}$
- B $A \cos 2x + B \sin 2x$
- C Ae^{2x}
- D $A + Bx$

3.2 Solve $\frac{dy}{dx} + \frac{1}{x}y = x^2$ (for $x > 0$), given $y = 1$ when $x = 1$. [5]

3.3 Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x$. [6]

Solutions

2.1 $\mu = e^{\int 3 dx} = e^{3x}$.

2.2 $m^2 - m - 6 = 0$.

3.1 B. Auxiliary $m^2 + 4 = 0 \Rightarrow m = \pm 2i$, giving $A \cos 2x + B \sin 2x$.

3.2 $\mu = e^{\int (1/x) dx} = e^{\ln x} = x$; $\frac{d}{dx}(xy) = x^3$; $xy = \frac{1}{4}x^4 + C$; $y = 1$ at $x = 1$: $1 = \frac{1}{4} + C \Rightarrow C = \frac{3}{4}$; $y = \frac{1}{4}x^3 + \frac{3}{4x}$.

3.3 auxiliary $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$, so CF = $Ae^x + Be^{2x}$; try PI $y = ax + b$: $-3a + 2(ax + b) = 4x \Rightarrow 2a = 4$, $-3a + 2b = 0$; $a = 2$, $b = 3$; general solution $y = Ae^x + Be^{2x} + 2x + 3$.