A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.6 Differential equations

Name: _____ Class: _____ Date: _____ **Total: 14 marks**

Objective

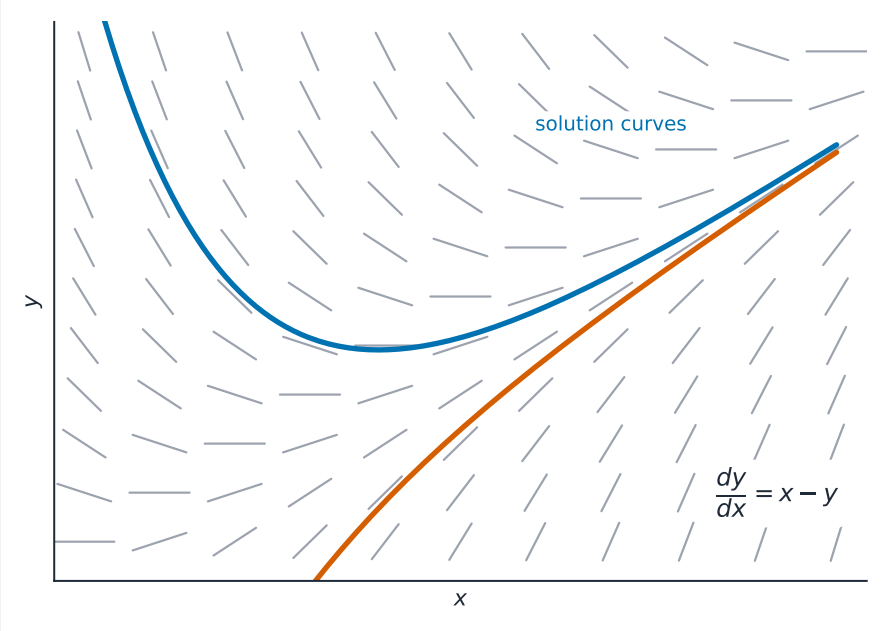
Build the skills to answer exam questions on **differential equations** 微分方程—the **integrating factor** 积分因子 for first-order linear equations, and the **complementary function** 余函数 plus **particular integral** 特积分 for second-order constant-coefficient equations.

You must be able to:

- solve $\dfrac{dy}{dx} + P(x)y = Q(x)$ with $\mu = e^{\int P\,dx}$
- solve $ay'' + by' + cy = 0$ via the auxiliary equation (the CF)
- find a particular integral and the general solution

1 Worked examples

■ Integrating factor & CF+PI



A solution curve stays tangent to the slope field everywhere

$\dfrac{dy}{dx} + 2y = e^x$**:** $\mu = e^{2x}$; $\dfrac{d}{dx}(ye^{2x}) = e^{3x}$, so $y = \frac{1}{3}e^x + Ce^{-2x}$.

> **Second order:** $y'' - 5y' + 6y = 0 \to$ auxiliary $m^2 - 5m + 6 = 0 \to y = Ae^{2x} + Be^{3x}$.

2 Practice

2.1 Write down the integrating factor for $\dfrac{dy}{dx} + 3y = x$. [1]

2.2 Write the auxiliary equation of $y'' - y' - 6y = 0$. [1]

3 Exam-style questions

3.1 The complementary function of $y'' + 4y = 0$ is: [1]

- **A** $Ae^{2x} + Be^{-2x}$
- **B** $A\cos 2x + B\sin 2x$
- **C** Ae^{2x}
- **D** $A + Bx$

3.2 Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2$ (for $x > 0$), given $y = 1$ when $x = 1$. [5]

3.3 Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x$. [6]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **2.6 Differential equations** past-paper sheet in the Library, or try these in **Practice mode**:

- 9231/21 N25 —Q4 (integrating factor)
- 9231/21 N25 —Q7 (second-order ODE)
- 9231/22 N25 —Q5 (differential equation)

Solutions

2.1 $\mu = e^{\int 3\,dx} = e^{3x}$.

2.2 $m^2 - m - 6 = 0$.

3.1 B. Auxiliary $m^2 + 4 = 0 \Rightarrow m = \pm 2i$, giving $A\cos 2x + B\sin 2x$.

3.2 $\mu = e^{\int (1/x)\,dx} = e^{\ln x} = x$; $\dfrac{d}{dx}(xy) = x^3$; $xy = \frac{1}{4}x^4 + C$; $y = 1$ at $x = 1$: $1 = \frac{1}{4} + C \Rightarrow C = \frac{3}{4}$; $y = \frac{1}{4}x^3 + \dfrac{3}{4x}$.

3.3 auxiliary $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$, so CF $= Ae^x + Be^{2x}$; try PI $y = ax + b$: $-3a + 2(ax + b) = 4x \Rightarrow 2a = 4,\ -3a + 2b = 0$; $a = 2,\ b = 3$; general solution $y = Ae^x + Be^{2x} + 2x + 3$.