A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.5 Complex numbers (de Moivre's theorem)

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

Objective

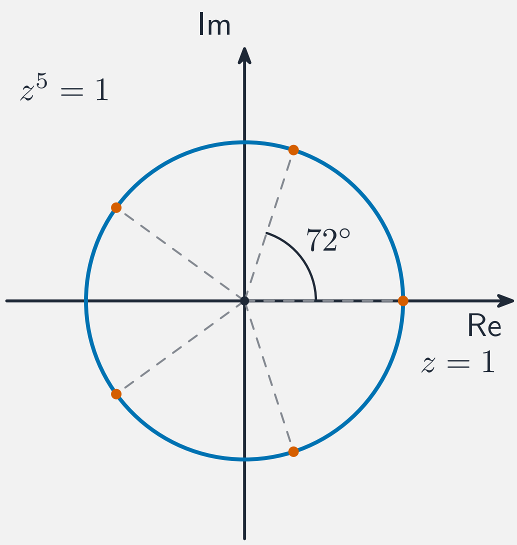
Build the skills to answer exam questions on **complex numbers 复数—de Moivre's theorem 棣莫弗定理**, expanding $\cos n\theta/\sin n\theta$, and the **roots of unity 单位根**.

You must be able to:

- use $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- express $\cos n\theta/\sin n\theta$ in powers (and powers in multiple angles)
- find the nth roots of a complex number

1 Worked examples

■ De Moivre & roots of unity



The solutions of $z^n = 1$ are equally spaced around the unit circle

$\cos 3\theta$**:** real part of $(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$.

Roots of unity: $z^n = 1 \Rightarrow z = e^{2\pi ik/n}$, $k = 0, \ldots, n-1$.

2 Practice

2.1 Use de Moivre to write $(\cos\theta + i\sin\theta)^4$ in the form $\cos(\,) + i\sin(\,)$. [1]

2.2 Write down the three cube roots of unity in polar form. [2]

3 Exam-style questions

3.1 By de Moivre's theorem, $(\cos\theta + i\sin\theta)^5$ equals: [1]

- **A** $\cos 5\theta + i\sin 5\theta$
- **B** $5\cos\theta + 5i\sin\theta$
- **C** $\cos\theta + i\sin 5\theta$
- **D** $\cos^5\theta + i\sin^5\theta$

3.2 (a) Use de Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. [4]

(b) Hence solve $4\sin^3\theta - 3\sin\theta + \frac{1}{2} = 0$ for $0 < \theta < \frac{\pi}{2}$. [3]

3.3 Find the four roots of $z^4 = 16$, giving them in the form $x + iy$. [4]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **2.5 Complex numbers** past-paper sheet in the Library, or try these in **Practice mode**:

- 9231/21 N25 —Q8 (de Moivre)
- 9231/22 N25 —Q2 (complex numbers)
- 9231/22 N25 —Q4 (roots of unity)

Solutions

2.1 $\cos 4\theta + i \sin 4\theta$.

2.2 $1,\ \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3},\ \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$.

3.1 **A**. $(\cos\theta + i \sin\theta)^5 = \cos 5\theta + i \sin 5\theta$.

3.2 (a) imaginary part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i \sin 3\theta$: $\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$; $= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$.
(b) $4\sin^3\theta - 3\sin\theta = -\sin 3\theta$, so the equation is $-\sin 3\theta + \frac{1}{2} = 0 \Rightarrow \sin 3\theta = \frac{1}{2}$; for $0 < 3\theta < \frac{3\pi}{2}$, $3\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; $\theta = \frac{\pi}{18}$ or $\frac{5\pi}{18}$.

3.3 $z^4 = 16 \Rightarrow z = 2e^{ik\pi/2}$, $k = 0, 1, 2, 3$; $z = 2,\ 2i,\ -2,\ -2i$.