2.4 Integration (standard forms, reduction formulae)

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS completing the square 配方法 • reduction formula 递推公式 • further integration 积分 • arc length 弧长

Objective

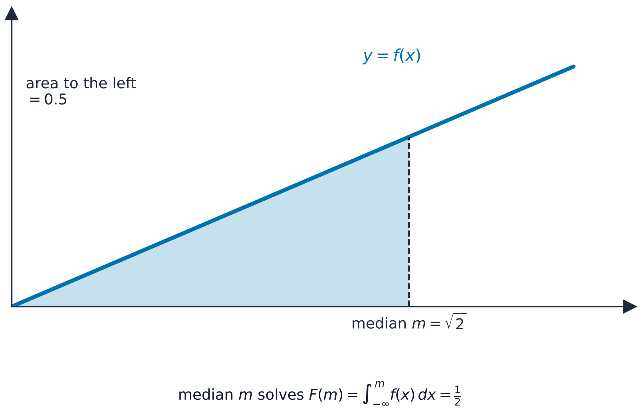
Build the skills to answer exam questions on **further integration 积分** — the standard inverse-function integrals, **reduction formulae 递推公式**, and **arc length 弧长**.

You must be able to:

- use $\int \frac{1}{a^2+x^2}\,dx$ and $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ (and completing the square)
- derive and use a reduction formula
- find an arc length

1 Worked examples

■ Standard integrals & reduction



An integral is the area under a curve; the inverse-function forms handle $\frac{1}{a^2+x^2}$ and $\frac{1}{\sqrt{a^2-x^2}}$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C.$$

Reduction: $I_n = \int_0^{\pi/2} \sin^n x\,dx$ satisfies $I_n = \dfrac{n-1}{n} I_{n-2}$.

2 Practice

2.1 Find $\displaystyle\int \frac{1}{9+x^2}\,dx$. [2]

2.2 Find $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$. [2]

3 Exam-style questions

3.1 $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$ equals: [1]

- **A** $\frac{\pi}{8}$
- **B** $\frac{\pi}{4}$
- **C** $\frac{\pi}{2}$
- **D** π

3.2 Let $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx$.

(a) Using integration by parts, show that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. [4]

(b) Hence evaluate I_4. [3]

3.3 Find $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$ by completing the square. [4]

Solutions

2.1 $\frac{1}{3}\tan^{-1}\dfrac{x}{3} + C$.

2.2 $\sin^{-1}\dfrac{x}{5} + C$.

3.1 A. $\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\tan^{-1}1 = \frac{1}{2}\cdot\frac{\pi}{4} = \dfrac{\pi}{8}$.

3.2 (a) $I_n = \int_0^{\pi/2}\sin^{n-1}x\sin x\,dx$; by parts ($u = \sin^{n-1}x$, $dv = \sin x\,dx$): $I_n = [-\sin^{n-1}x\cos x]_0^{\pi/2} + (n-1)\int_0^{\pi/2}\sin^{n-2}x\cos^2 x\,dx$; the boundary term is 0; $\cos^2 x = 1 - \sin^2 x$ gives $I_n = (n-1)(I_{n-2} - I_n)$; so $nI_n = (n-1)I_{n-2}$, i.e. $I_n = \dfrac{n-1}{n}I_{n-2}$.
(b) $I_0 = \dfrac{\pi}{2}$; $I_2 = \frac{1}{2}I_0 = \dfrac{\pi}{4}$; $I_4 = \frac{3}{4}I_2 = \dfrac{3\pi}{16}$.

3.3 $x^2 - 4x + 13 = (x-2)^2 + 9$; $\displaystyle\int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C$.