A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.3 Differentiation (Maclaurin, hyperbolic, parametric)

Name: ____________ Class: ________ Date: __________ **Total: 37 marks**

KEY WORDS maclaurin's series 麦克劳林级数 • hyperbolic functions 双曲函数 • further differentiation 微分

Objective

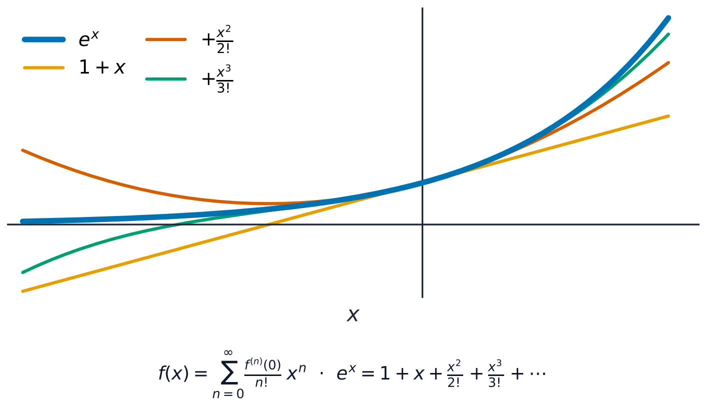
Build the skills to answer exam questions on **further differentiation 微分** — derivatives of inverse and **hyperbolic functions** 双曲函数, second derivatives of implicit/parametric curves, and **Maclaurin's series** 麦克劳林级数.

You must be able to:

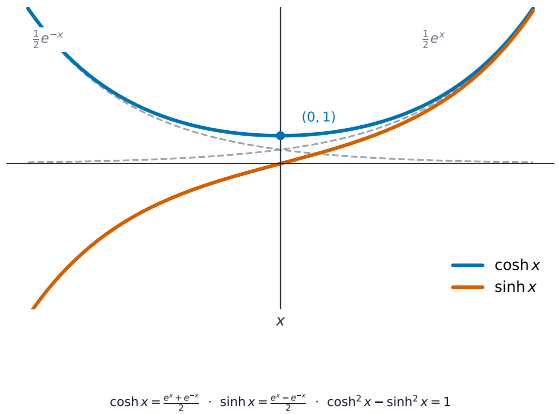
- differentiate $\sin^{-1} x$, $\tan^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$
- find $\dfrac{d^2y}{dx^2}$ for implicit/parametric curves
- find a Maclaurin series for a function

1 Worked examples

■ Maclaurin series



Adding more terms makes the polynomial match the function over a wider stretch around $x = 0$



The hyperbolic functions whose derivatives appear here

Maclaurin: $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots$; e.g. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots$.

Derivatives: $\dfrac{d}{dx}\sinh x = \cosh x$; $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1 + x^2}$.

■ **The derivatives to know, and the substitutions they unlock**

Hyperbolic functions, defined from the exponential:

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

The identity is $\cosh^2 x - \sinh^2 x = 1$ — note the **minus**, which is what makes the substitutions differ from the trigonometric ones.

y	$\dfrac{dy}{dx}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$ (no minus — unlike cos)
$\tanh x$	$\operatorname{sech}^2 x$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\operatorname{arsinh} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\dfrac{1}{\sqrt{x^2-1}}$

Reading the integral to choose the substitution — the sign under the root decides:

Integrand contains	**Substitute**	**Because**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2=\cos^2$
a^2+x^2	$x = a\tan\theta$	$1+\tan^2=\sec^2$
$\sqrt{a^2+x^2}$	$x = a\sinh u$	$1+\sinh^2=\cosh^2$
$\sqrt{x^2-a^2}$	$x = a\cosh u$	$\cosh^2-1=\sinh^2$

Worked example. Differentiate $y = \operatorname{arsinh}(3x)$ and hence write down $\displaystyle\int \frac{1}{\sqrt{1+9x^2}}\,dx$.

1. By the chain rule, $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{1+(3x)^2}} = \dfrac{3}{\sqrt{1+9x^2}}$.
2. So $\displaystyle\int \frac{3}{\sqrt{1+9x^2}}\,dx = \operatorname{arsinh}(3x) + C$.
3. Dividing by 3: $\displaystyle\int \frac{1}{\sqrt{1+9x^2}}\,dx = \tfrac{1}{3}\operatorname{arsinh}(3x) + C$.

That is the pattern behind almost every "differentiate, hence integrate" question: the integral you are asked for is the derivative you just found, scaled.

2 Practice

2.1 Differentiate $y = \cosh(3x)$. [2]

2.2 Differentiate $y = \sin^{-1}(2x)$. [2]

2.3 Differentiate $y = \cosh(4x)$ and $y = \arctan(2x)$. [3]

2.4 State the substitution you would use for $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$, and why. [2]

2.5 Show that $\cosh^2 x - \sinh^2 x = 1$ using the exponential definitions. [3]

3 Exam-style questions

3.1 $\dfrac{d}{dx}\tanh x$ equals: [1]

- **A** $\operatorname{sech}^2 x$
- **B** $\operatorname{cosech}^2 x$
- **C** $\cosh^2 x$
- **D** $\sinh^2 x$

3.2 Find the Maclaurin series for $\ln(1+x)$ up to and including the term in x^3. [4]

3.3 A curve is given by $x = t^2$, $y = t^3$. Find $\dfrac{d^2y}{dx^2}$ in terms of t. [5]

3.4 (a) Differentiate $y = \operatorname{arcosh}(2x)$ with respect to x. [3]

(b) Hence find $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}\,dx$. [2]

(c) Use the substitution $x = 3\sinh u$ to show that $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{3}\right) + C$. [5]

(d) Differentiate $y = x\arctan x$ and hence evaluate $\displaystyle\int_0^1 \arctan x\,dx$, giving your answer in exact form. [5]

Solutions

2.1 $\dfrac{dy}{dx} = 3\sinh(3x)$.

2.2 $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-4x^2}}$.

2.3 $\dfrac{d}{dx}\cosh 4x = 4\sinh 4x$; $\dfrac{d}{dx}\arctan 2x = \dfrac{2}{1+4x^2}$.

2.4 $x = 3\cosh u$; because $\cosh^2 u - 1 = \sinh^2 u$ removes the root when $a = 3$.

2.5 $\cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$; $\sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$; subtracting, $\frac{1}{4}(4) = 1$.

3.1 **A**. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$.

3.2 $f(0) = 0$; $f'(x) = \dfrac{1}{1+x}$, $f'(0) = 1$; $f''(x) = -\dfrac{1}{(1+x)^2}$, $f''(0) = -1$; $f'''(0) = 2$; $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$.

3.3 $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$, so $\dfrac{dy}{dx} = \dfrac{3t^2}{2t} = \dfrac{3t}{2}$; $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{3t}{2}\right) \div \dfrac{dx}{dt} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$.

3.4 (a) $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{(2x)^2-1}} = \dfrac{2}{\sqrt{4x^2-1}}$.

(b) The integrand is half the derivative in (a), so the integral is $\frac{1}{2}\operatorname{arcosh}(2x) + C$.

(c) $x = 3\sinh u$ so $\dfrac{dx}{du} = 3\cosh u$; $9 + x^2 = 9(1+\sinh^2 u) = 9\cosh^2 u$, so $\sqrt{9+x^2} = 3\cosh u$; the integral becomes $\displaystyle\int \frac{3\cosh u}{3\cosh u}\,du = \int 1\,du = u + C$; and $u = \operatorname{arsinh}(x/3)$.

(d) $\dfrac{d}{dx}(x\arctan x) = \arctan x + \dfrac{x}{1+x^2}$; integrating both sides from 0 to 1, $[x\arctan x]_0^1 = \displaystyle\int_0^1 \arctan x\,dx + \left[\tfrac{1}{2}\ln(1+x^2)\right]_0^1$; so $\dfrac{\pi}{4} = I + \frac{1}{2}\ln 2$; $I = \dfrac{\pi}{4} - \frac{1}{2}\ln 2$.