A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.2 Matrices (eigenvalues and eigenvectors)

Name: ____________ Class: ________ Date: __________ **Total: 34 marks**

KEY WORDS characteristic equation 特征方程 • eigenvalue 特征值 • eigenvector 特征向量 • matrices 矩阵

Objective

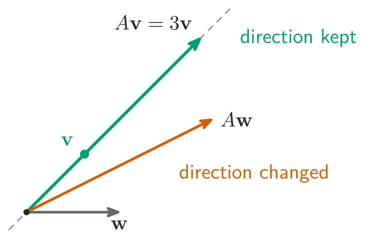
Build the skills to answer exam questions on **matrices** 矩阵 — solving $A\mathbf{x} = \mathbf{b}$, the **characteristic equation** 特征方程, **eigenvalues** 特征值 and **eigenvectors** 特征向量, and diagonalisation.

You must be able to:

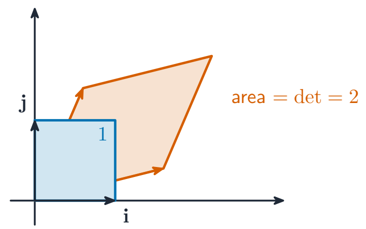
- find eigenvalues from $\det(A - \lambda I) = 0$
- find an eigenvector for each eigenvalue
- write $A = QDQ^{-1}$ and use it (e.g. for A^n)

1 Worked examples

Eigenvalues & eigenvectors



Multiplying an eigenvector by A only stretches it (by λ); a general vector changes direction



Eigenvectors are the directions a matrix leaves unturned

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$**:** $\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 1$ or 3. Eigenvector for $\lambda = 3$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; for $\lambda = 1$: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Eigenvalues, eigenvectors, and what diagonalisation buys you

An **eigenvector** 特征向量 of A is a non-zero $\mathbf{v}$ that A only stretches: $A\mathbf{v} = \lambda\mathbf{v}$, where λ is the **eigenvalue** 特征值. The direction survives the transformation; only the length changes.

The method, which is the same every time:

1. **Characteristic equation:** $\det(A - \lambda I) = 0$. For a 2×2 this is $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$, which is worth knowing as a check.
2. **Solve** for the eigenvalues.
3. **For each** λ, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$. The rows are dependent by construction, so you get a **family** — quote any convenient member, and if the rows are NOT dependent you have made an arithmetic slip.
4. **Diagonalise:** put the eigenvectors in the columns of P and the matching eigenvalues down the diagonal of D, in the **same order**. Then $A = PDP^{-1}$.

Why it is worth doing: $A^n = PD^nP^{-1}$, and D^n is just each diagonal entry raised to n. Computing A^{10} directly is ten matrix multiplications; this is one.

The Cayley–Hamilton theorem says a matrix satisfies its own characteristic equation. For a 2×2 with $\lambda^2 - t\lambda + d = 0$, that gives $A^2 = tA - dI$ — which reduces any power of A, and rearranges to $A^{-1} = \frac{1}{d}(tI - A)$.

Worked example. For $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, find the eigenvalues and one eigenvector for each.

1. $\operatorname{tr} A = 7$, $\det A = 10$, so $\lambda^2 - 7\lambda + 10 = 0$.
2. $(\lambda - 5)(\lambda - 2) = 0$, so $\lambda = 5$ or $\lambda = 2$.
3. For $\lambda = 5$: $(A - 5I)\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0}$ gives $v_1 = v_2$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. For $\lambda = 2$: $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$ gives $2v_1 = -v_2$, so $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

2 Practice

2.1 Write down the characteristic equation of a matrix A. [1]

2.2 Find the eigenvalues of $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$. [1]

2.3 Write down the characteristic equation of $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ using its trace and determinant. [2]

2.4 State what P and D contain in the diagonalisation $A = PDP^{-1}$. [2]

2.5 Use the Cayley–Hamilton theorem to express A^2 in terms of A and I for a matrix with $\lambda^2 - 6\lambda + 8 = 0$. [2]

3 Exam-style questions

3.1 A vector $\mathbf{v}$ is an eigenvector of A with eigenvalue λ when: [1]

- **A** $A\mathbf{v} = \mathbf{0}$
- **B** $A\mathbf{v} = \lambda\mathbf{v}$
- **C** $\det A = \lambda$
- **D** $A = \lambda I$

3.2 The matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find the eigenvalues. [2]

(b) Find an eigenvector for each eigenvalue. [4]

3.3 The matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1 and 3. Using $M = QDQ^{-1}$, find M^2 and verify by direct multiplication. [5]

3.4 The matrix $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$.

(a) Show that the characteristic equation of A is $\lambda^2 - 7\lambda + 6 = 0$. [3]

(b) Find the eigenvalues of A. [2]

(c) Find an eigenvector corresponding to each eigenvalue. [4]

(d) Write down matrices P and D such that $A = PDP^{-1}$. [2]

(e) Explain why this makes A^n easy to compute, and state the form of A^n. [3]

Solutions

2.1 $\det(A - \lambda I) = 0$.

2.2 $\lambda = 4$ and $\lambda = -2$ (the diagonal entries).

2.3 $\operatorname{tr} A = 7$ and $\det A = 10$, so $\lambda^2 - 7\lambda + 10 = 0$.

2.4 P has the eigenvectors as its columns; D is diagonal with the matching eigenvalues, in the same order.

2.5 $A^2 - 6A + 8I = \mathbf{0}$, so $A^2 = 6A - 8I$.

3.1 B. $A\mathbf{v} = \lambda\mathbf{v}$ defines an eigenvector.

3.2 (a) $\det \begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 3, 2$.

(b) $\lambda = 3$: $(A - 3I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda = 2$: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3.3 $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, so $M^2 = QD^2Q^{-1}$ with $D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$; this gives $M^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$; direct: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ ✓.

3.4 (a) $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6$, and setting it to zero gives the result.

(b) $(\lambda - 6)(\lambda - 1) = 0$, so $\lambda = 6$ or $\lambda = 1$.

(c) For $\lambda = 6$: $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\mathbf{v} = \mathbf{0}$ gives $v_1 = 2v_2$, so $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For $\lambda = 1$: $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0}$ gives $2v_1 = -v_2$, so $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(d) $P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$.

(e) $A^n = PD^nP^{-1}$; D^n is found by raising each diagonal entry to the power n, so $A^n = P\begin{pmatrix} 6^n & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$, one multiplication instead of n.