2.2 Matrices (eigenvalues and eigenvectors)

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

Objective

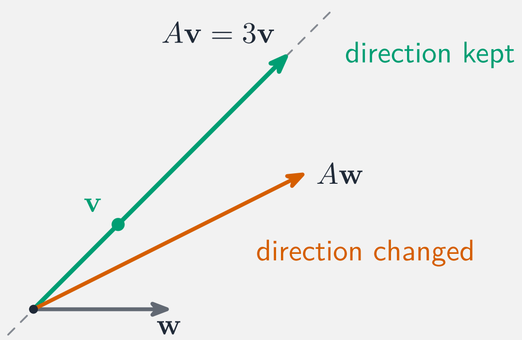
Build the skills to answer exam questions on **matrices** 矩阵—solving $A\mathbf{x} = \mathbf{b}$, the **characteristic equation** 特征方程, **eigenvalues** 特征值 and **eigenvectors** 特征向量, and diagonalisation.

You must be able to:

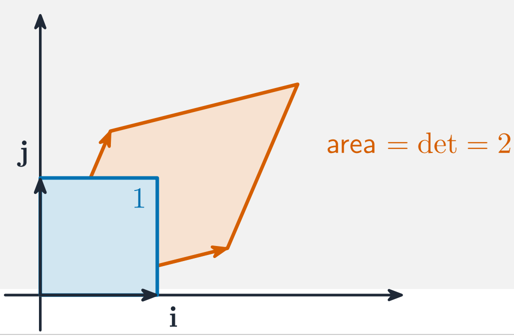
- find eigenvalues from $\det(A - \lambda I) = 0$
- find an eigenvector for each eigenvalue
- write $A = QDQ^{-1}$ and use it (e.g. for A^n)

1 Worked examples

■ Eigenvalues & eigenvectors



Multiplying an eigenvector by A only stretches it (by λ); a general vector changes direction



Eigenvectors are the directions a matrix leaves unturned

> $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$**:** $\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 1$ or 3. Eigenvector for $\lambda = 3$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; for $\lambda = 1$: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2 Practice

2.1 Write down the characteristic equation of a matrix A. [1]

2.2 Find the eigenvalues of $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$. [1]

3 Exam-style questions

3.1 A vector $\mathbf{v}$ is an eigenvector of A with eigenvalue λ when: [1]

- **A** $A\mathbf{v} = \mathbf{0}$
- **B** $A\mathbf{v} = \lambda\mathbf{v}$
- **C** $\det A = \lambda$
- **D** $A = \lambda I$

3.2 The matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find the eigenvalues. [2]

(b) Find an eigenvector for each eigenvalue. [4]

3.3 The matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1 and 3. Using $M = QDQ^{-1}$, find M^2 and verify by direct multiplication. [5]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **2.2 Matrices** past-paper sheet in the Library, or try these in **Practice mode**:

- 9231/21 N25 —Q1 (eigenvalues)
- 9231/21 N25 —Q6 (eigenvectors / diagonalisation)
- 9231/22 N25 —Q8 (matrices)

Solutions

2.1 $\det(A - \lambda I) = 0$.

2.2 $\lambda = 4$ and $\lambda = -2$ (the diagonal entries).

3.1 B. $A\mathbf{v} = \lambda\mathbf{v}$ defines an eigenvector.

3.2 (a) $\det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 3, 2$.

(b) $\lambda = 3$: $(A - 3I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda = 2$: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3.3 $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, so $M^2 = QD^2Q^{-1}$ with $D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$; this gives $M^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$; direct: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.