A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

1.7 Proof by induction

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS proof by induction 数学归纳法 • inductive proof 归纳证明

Objective

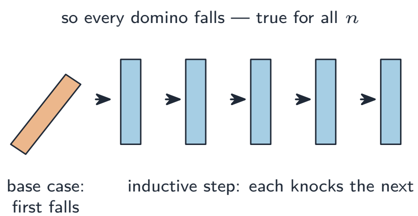
Build the skills to answer exam questions on **proof by induction 数学归纳法** — the base case, the inductive step, and a clear concluding statement, for series, divisibility and matrices.

You must be able to:

- prove a summation formula by induction
- prove a divisibility result by induction
- write the base case, inductive step and conclusion clearly

1 Worked examples

■ The two steps



Base case topples the first; the inductive step makes each one knock over the next

Prove $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$**.** Base $n = 1$: $1 = \frac{1}{2}(1)(2)$ ✓. Assume true for k; then $\sum_{1}^{k+1} = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$ — the formula at $n = k+1$. So true for all n.

2 Practice

2.1 State the two steps of a proof by induction. [2]

2.2 Verify the base case $n = 1$ for $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [1]

3 Exam-style questions

3.1 In an inductive proof, the inductive step assumes the result for $n = k$ and proves it for: [1]

- **A** $n = 1$
- **B** $n = k - 1$
- **C** $n = k + 1$
- **D** all n at once

3.2 Prove by induction that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [5]

3.3 Prove by induction that $3^{2n} - 1$ is divisible by 8 for all positive integers n. [5]

Solutions

2.1 base case (true for $n = 1$); inductive step (assume $n = k$, prove $n = k + 1$).

2.2 LHS $= 1$, RHS $= \frac{1}{6}(1)(2)(3) = 1$ ✓.

3.1 **C**. The inductive step proves the result for $n = k + 1$.

3.2 base $n = 1$: LHS $= 2$, RHS $= \frac{1}{3}(1)(2)(3) = 2$ ✓; assume $\sum_1^k r(r+1) = \frac{1}{3}k(k+1)(k+2)$; $\sum_1^{k+1} = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$; $= (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{1}{3}(k+1)(k+2)(k+3)$; this is the formula at $n = k + 1$, so true for all n.

3.3 base $n = 1$: $3^2 - 1 = 8$, divisible by 8 ✓; assume $3^{2k} - 1 = 8m$ for some integer m, so $3^{2k} = 8m + 1$; then $3^{2(k+1)} - 1 = 3^2 \cdot 3^{2k} - 1 = 9(8m + 1) - 1 = 72m + 8$; $= 8(9m + 1)$, and $9m + 1$ is an integer; so the result holds at $n = k + 1$ whenever it holds at $n = k$, and by induction for all positive n.