A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

1.7 Proof by induction

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

Objective

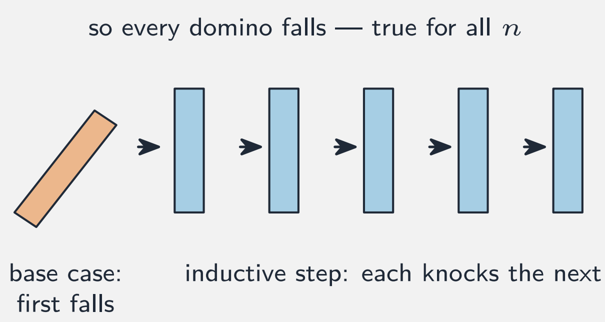
Build the skills to answer exam questions on **proof by induction** 数学归纳法—the base case, the inductive step, and a clear concluding statement, for series, divisibility and matrices.

You must be able to:

- prove a summation formula by induction
- prove a divisibility result by induction
- write the base case, inductive step and conclusion clearly

1 Worked examples

■ The two steps



Base case topples the first; the inductive step makes each one knock over the next

Prove $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$**.** Base $n = 1$: $1 = \frac{1}{2}(1)(2)$. Assume true for k; then $\sum_{1}^{k+1} = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$ —the formula at $n = k+1$. So true for all n.

2 Practice

2.1 State the two steps of a proof by induction. [2]

2.2 Verify the base case $n = 1$ for $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [1]

3 Exam-style questions

3.1 In an inductive proof, the inductive step assumes the result for $n = k$ and proves it for: [1]

- **A** $n = 1$
- **B** $n = k - 1$
- **C** $n = k + 1$
- **D** all n at once

3.2 Prove by induction that $\displaystyle\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [5]

3.3 Prove by induction that $7^n - 1$ is divisible by 6 for all positive integers n. [5]

4 Go further

You are now ready for the real exam questions on this subtopic. Open the **1.7 Proof by induction** past-paper sheet in the Library, or try these in **Practice mode**:

- 9231/11 N25 —Q3 (induction)
- 9231/12 N25 —Q3 (induction)
- 9231/14 N25 —Q1 (induction)

Solutions

2.1 base case (true for $n = 1$); inductive step (assume $n = k$, prove $n = k + 1$).

2.2 LHS $= 1$, RHS $= \frac{1}{6}(1)(2)(3) = 1$.

3.1 **C**. The inductive step proves the result for $n = k + 1$.

3.2 base $n = 1$: LHS $= 2$, RHS $= \frac{1}{3}(1)(2)(3) = 2$; assume $\sum_1^k r(r+1) = \frac{1}{3}k(k+1)(k+2)$; $\sum_1^{k+1} = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$; $= (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{1}{3}(k+1)(k+2)(k+3)$; this is the formula at $n = k + 1$, so true for all n.

3.3 base $n = 1$: $7^1 - 1 = 6$, divisible by 6 ; assume $7^k - 1 = 6m$ for integer m; $7^{k+1} - 1 = 7 \cdot 7^k - 1 = 7(6m + 1) - 1 = 42m + 6 = 6(7m + 1)$; divisible by 6, so by induction true for all n.