1.6 Vectors

Name: ____________ Class: ________ Date: __________ **Total: 18 marks**

KEY WORDS plane 平面 • vectors 向量 • vector product 向量积

Objective

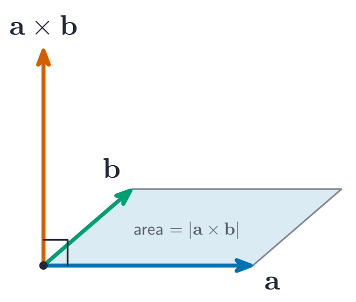
Build the skills to answer exam questions on **vectors 向量** in 3-D — the **vector product 向量积**, equations of a **plane 平面**, and distances/angles between lines and planes.

You must be able to:

- compute $\mathbf{a} \times \mathbf{b}$ and use it for a normal/area
- write a plane as $\mathbf{r} \cdot \mathbf{n} = p$ or $ax + by + cz = d$
- find the angle between, and distance from, lines and planes

1 Worked examples

■ **Vector product & plane**



$\mathbf{a} \times \mathbf{b}$ *is perpendicular to both; its length is the area of the parallelogram*

$\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$: $\mathbf{a} \times \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

Plane through A with normal $\mathbf{n}$: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

2 Practice

2.1 Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$. [3]

2.2 Write the plane through $(1, 2, 3)$ with normal $\mathbf{n} = (1, 1, 1)$ in the form $ax + by + cz = d$. [2]

3 Exam-style questions

3.1 $\mathbf{a} \times \mathbf{b}$ is a vector that is: [1]

- **A** parallel to $\mathbf{a}$
- **B** perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- **C** equal to $\mathbf{a} \cdot \mathbf{b}$
- **D** of length $|\mathbf{a}||\mathbf{b}| \cos\theta$

3.2 Points $P(1,0,2)$, $Q(3,1,1)$ and $R(2,-2,4)$ are given.

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$, and hence the area of triangle PQR. [4]

(b) Find the equation of the plane PQR, and the perpendicular distance from the origin to it. [4]

3.3 Find the acute angle between the planes $x+2y+2z=5$ and $2x-y+2z=1$. [4]

Solutions

2.1 $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(2) + \mathbf{k}(-1) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

2.2 $x+y+z = 1+2+3 = 6$.

3.1 B. The vector product is perpendicular to both vectors.

3.2 (a) $\overrightarrow{PQ} = (2,1,-1)$, $\overrightarrow{PR} = (1,-2,2)$; cross $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = \mathbf{i}(2-2) - \mathbf{j}(4+1) + \mathbf{k}(-4-1) = (0,-5,-5)$; the triangle is half the parallelogram, so its area is $\frac{1}{2}|(0,-5,-5)| = \frac{1}{2} \cdot 5\sqrt{2} = \frac{5\sqrt{2}}{2}$.
(b) any parallel normal will do, so take $(0,1,1)$; through $P(1,0,2)$ this gives $y+z=2$. Distance from the origin $= \dfrac{|0+0-2|}{\sqrt{0^2+1^2+1^2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

3.3 normals $(1,2,2)$ and $(2,-1,2)$; dot $= 2-2+4 = 4$; $|\mathbf{n}_1| = 3$, $|\mathbf{n}_2| = 3$; $\cos\theta = \dfrac{4}{9} \Rightarrow \theta = 63.6°$.