Past-paper walk-through

Probability generating functions ▪ 4.5

eXercise

Questions in this set

- 9231/41 N25 Q7 ▪ 10 marks
- 9231/42 N25 Q6 ▪ 10 marks
- 9231/44 N25 Q4 ▪ 10 marks
- 9231/41 J25 Q6 ▪ 6 marks
- 9231/43 J25 Q6 ▪ 11 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 7

9231/41 N25 ▪ Q7 ▪ 10 marks

A discrete random variable X takes values $r = 0, 1, 2$ with probabilities $\mathrm{P}(X = r)$ given by $\mathrm{P}(X = 0) = a$, $\mathrm{P}(X = 1) = 2a$, $\mathrm{P}(X = 2) = b$.

(a) Write down the probability generating function of X, and use it to find an expression for $\mathrm{E}(X)$ in terms of a and b. **[2]**

(b) Show that $\mathrm{Var}(X) = 2b + 2(a + b)(1 - 2a - 2b)$. **[3]**

(c) The random variable Y is defined by $Y = X_1 + X_2 + X_3 + \cdots + X_{10}$ where $X_1, X_2, X_3, \ldots, X_{10}$ are ten independent observations of X. Using the probability generating function of Y, and your answer to part (a), show that $\mathrm{E}(Y) = 10\mathrm{E}(X)$. **[3]**

(d) For the case $b = 0$, define fully the distribution of Y. **[2]**

Question 7

r	0	1	2
$\mathrm{P}(X = r)$	a	$2a$	b

Solution 7

(a)

Mark scheme

- $\mathrm{G}_X(t) = a + 2at + bt^2$. $\mathrm{G}'_X(t) = 2a + 2bt$, so $\mathrm{E}(X) = \mathrm{G}'_X(1) = 2a + 2b$.

Solution 7

(b)

Mark scheme

- $\mathrm{G}_X''(t) = 2b$, so $\mathrm{G}_X''(1) = 2b$.
  $\mathrm{Var}(X) = \mathrm{G}_X''(1) + \mathrm{G}_X'(1) - [\mathrm{G}_X'(1)]^2 = 2b + (2a + 2b) - (2a + 2b)^2 =$
  $2b + (2a + 2b)(1 - (2a + 2b)) = 2b + 2(a + b)(1 - 2a - 2b)$ (AG).

Solution 7

(c)

Mark scheme

- $\mathrm{G}_Y(t) = [\mathrm{G}_X(t)]^{10} = (a + 2at + bt^2)^{10}$. $\mathrm{G}'_Y(t) = 10(a + 2at + bt^2)^9(2a + 2bt)$, so $\mathrm{E}(Y) = \mathrm{G}'_Y(1) = 10(3a + b)^9(2a + 2b)$. Since the total probability is $3a + b = 1$, $\mathrm{E}(Y) = 10(2a + 2b) = 10\mathrm{E}(X)$.

Solution 7

(d)

Mark scheme

- When $b = 0$, $3a = 1$ so $a = \frac{1}{3}$: X takes value 0 with probability $\frac{1}{3}$ and value 1 with probability $\frac{2}{3}$ (a Bernoulli trial). Hence Y is $\mathrm{B}\left(10, \frac{2}{3}\right)$.

Question 6

9231/42 N25 ▪ Q6 ▪ 10 marks

The discrete random variable X has probability generating function $G_X(t) = \dfrac{t}{(3-2t)^2}$.

(a) Find $E(X)$ and $\mathrm{Var}(X)$. **[5]**

(b) The discrete random variable Y has probability generating function $G_Y(t) = \dfrac{t^2}{(3-2t)^2}$. The random variable Z is the sum of X and Y. Assuming X and Y are independent, find $P(Z > 4)$. **[5]**

Solution 6

(a)

Mark scheme

- $G_X'(t) = (3-2t)^{-2} + 4t(3-2t)^{-3} \Rightarrow E(X) = G_X'(1) = 5$. $G_X''(1) = 32$, so $\mathrm{Var}(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2 = 32 + 5 - 25 = 12$.

Solution 6

(b) Mark scheme

- $G_Z(t) = G_X(t)G_Y(t) = \dfrac{t^3}{(3-2t)^4}$. Then $P(Z=3) = \frac{1}{81}$ and $P(Z=4) = \frac{8}{243}$, so $P(Z>4) = 1 - \frac{11}{243} = \frac{232}{243} \approx 0.955$.

Question 4

9231/44 N25 ▪ Q4 ▪ 10 marks

The random variable X takes values 1 and 2 with probabilities $\frac{2}{5}$ and $\frac{3}{5}$ respectively.

(a) Write down the probability generating function $\mathrm{G}_X(t)$ of X. **[1]**

(b) The random variable Y is the sum of four independent observations of X. Find the probability generating function $\mathrm{G}_Y(t)$ of Y. Give your answer in the form $\mathrm{G}_Y(t) = at^m(b + ct)^n$, where a, b, c, m and n are constants to be determined. **[2]**

(c) Use $\mathrm{G}_Y(t)$ to find $\mathrm{P}(Y = 6)$. **[2]**

(d) Find $\mathrm{Var}(Y)$. **[5]**

Solution 4

(a)

Mark scheme

- $\mathrm{G}_X(t) = \frac{2}{5}t + \frac{3}{5}t^2.$

Solution 4

(b)

Mark scheme

- $\mathrm{G}_Y(t) = [\mathrm{G}_X(t)]^4 = \left(\frac{2}{5}t + \frac{3}{5}t^2\right)^4 = \frac{1}{625}t^4(2+3t)^4.$

Solution 4

(c)

Mark scheme

- $\mathrm{P}(Y=6)$ is the coefficient of t^6 in $\mathrm{G}_Y(t)$, i.e. $\frac{1}{625} \times \binom{4}{2} 2^2 3^2 = \frac{216}{625} \approx 0.346$.

Solution 4

(d) Mark scheme

- $\mathrm{E}(X) = \frac{8}{5}$, $\mathrm{E}(X^2) = \frac{14}{5}$, so $\mathrm{Var}(X) = \frac{14}{5} - \left(\frac{8}{5}\right)^2 = \frac{6}{25}$. Hence $\mathrm{Var}(Y) = 4\,\mathrm{Var}(X) = \frac{24}{25} = 0.96$. (Via the PGF: $\mathrm{G}'_Y(1) = \frac{32}{5} = 6.4$, $\mathrm{G}''_Y(1) = \frac{888}{25} = 35.52$, $\mathrm{Var}(Y) = \frac{888}{25} + \frac{32}{5} - \left(\frac{32}{5}\right)^2 = \frac{24}{25}$.)

Question 6

9231/41 J25 ▪ Q6 ▪ 6 marks

Y is a discrete random variable which takes the values $0, 2, 4, \ldots$. The probability generating function of Y is given by $\mathrm{G}_Y(t) = \dfrac{k}{1 - at^2}$.

(a) Find k in terms of a. **[1]**

(b) Show that $\mathrm{P}(Y > 2) = a^2$. **[3]**

(c) It is now given that $a = 0.2$. Find the value of $\mathrm{E}(Y)$. **[2]**

Solution 6

(a)

Mark scheme

- $\mathrm{G}_Y(1) = 1$ gives $\dfrac{k}{1-a} = 1$, so $k = 1 - a$.

Solution 6

(b)

Mark scheme

- Expanding $(1 - at^2)^{-1} = 1 + at^2 + \cdots$, so $\mathrm{P}(Y = 0) = k$ and $\mathrm{P}(Y = 2) = ka$. Hence $\mathrm{P}(Y > 2) = 1 - k(1 + a) = 1 - (1 - a)(1 + a) = 1 - (1 - a^2) = a^2$ (AG).

Solution 6

(c)

Mark scheme

- $\mathrm{G}'_Y(t) = \dfrac{2akt}{(1-at^2)^2} = \dfrac{0.32t}{(1-0.2t^2)^2}$ (with $a = 0.2$, $k = 0.8$). So $\mathrm{E}(Y) = \mathrm{G}'_Y(1) = \dfrac{0.32}{(1-0.2)^2} = \dfrac{0.32}{0.64} = 0.5.$

Question 6

9231/43 J25 ▪ Q6 ▪ 11 marks

A bag contains 7 red balls and 3 blue balls. Kieran selects 2 balls at random, without replacement. The number of red balls selected by Kieran is denoted by X, and the number of different colours present in Kieran's selection is denoted by Y.

(a) Find the probability generating functions $\mathrm{G}_X(t)$ of X and $\mathrm{G}_Y(t)$ of Y. **[4]**

(b) The random variable Z is the sum of the number of red balls and the number of different colours present in Kieran's selection. Kieran claims that the probability generating function of Z is equal to $\mathrm{G}_X(t) \times \mathrm{G}_Y(t)$. Explain why Kieran is incorrect. **[1]**

(c) Find the probability generating function of Z, expressing your answer as a polynomial in t. **[4]**

(d) Use the probability generating function of Z to find $\mathrm{E}(Z)$. **[2]**

Solution 6

(a) Mark scheme

- X (number of red balls): $\mathrm{P}(X=0)=\frac{1}{15}$, $\mathrm{P}(X=1)=\frac{7}{15}$, $\mathrm{P}(X=2)=\frac{7}{15}$, so $\mathrm{G}_X(t)=\frac{1}{15}+\frac{7}{15}t+\frac{7}{15}t^2$. Y (number of different colours): $\mathrm{P}(Y=1)=\frac{8}{15}$, $\mathrm{P}(Y=2)=\frac{7}{15}$, so $\mathrm{G}_Y(t)=\frac{8}{15}t+\frac{7}{15}t^2$.

Solution 6

(b)

Mark scheme

- X and Y are not independent, so the PGF of $Z = X + Y$ is not the product $\mathrm{G}_X(t)\,\mathrm{G}_Y(t)$.

Solution 6

(c) Mark scheme

- The non-zero cases are $(X, Y) = (0, 1)$ giving $Z = 1$ with $\mathrm{P} = \frac{1}{15}$; $(1, 2)$ giving $Z = 3$ with $\mathrm{P} = \frac{7}{15}$; and $(2, 1)$ giving $Z = 3$ with $\mathrm{P} = \frac{7}{15}$. So $\mathrm{P}(Z = 1) = \frac{1}{15}$, $\mathrm{P}(Z = 3) = \frac{14}{15}$, and $\mathrm{G}_Z(t) = \frac{1}{15}t + \frac{14}{15}t^3$.

Solution 6

(d)

Mark scheme

- $\mathrm{G}'_Z(t) = \frac{1}{15} + \frac{42}{15}t^2$, so $\mathrm{E}(Z) = \mathrm{G}'_Z(1) = \frac{1}{15} + \frac{42}{15} = \frac{43}{15} \approx 2.87$.