Past-paper walk-through

Continuous random variables ▪ 4.1

eXercise

Questions in this set

- 9231/41 N25 Q5 ▪ 10 marks
- 9231/42 N25 Q4 ▪ 10 marks
- 9231/44 N25 Q2 ▪ 10 marks
- 9231/41 J25 Q2 ▪ 12 marks
- 9231/43 J25 Q3 ▪ 10 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 5

9231/41 N25 ▪ Q5 ▪ 10 marks

A continuous random variable X has probability density function f given by $\mathrm{f}(x) = \frac{1}{16}\sqrt{x}$ for $0 \leqslant x < 4$, $\dfrac{1}{k\sqrt{x}}$ for $4 \leqslant x \leqslant 9$, and 0 otherwise, where k is a constant.

(a) Show that $k = 3$. **[2]**

(b) Find the median value of X. **[3]**

(c) The random variable Y is defined by $Y = \sqrt{X}$. Find the probability density function of Y. **[5]**

Solution 5

(a)

Mark scheme

- $\int_0^4 \frac{1}{16}\sqrt{x}\,dx + \int_4^9 \frac{1}{k\sqrt{x}}\,dx = 1$: $\frac{1}{16}\left[\frac{2}{3}x^{3/2}\right]_0^4 + \frac{1}{k}\left[2\sqrt{x}\right]_4^9 = \frac{1}{3} + \frac{2}{k} = 1$, so $\frac{2}{k} = \frac{2}{3}$, giving $k = 3$ (AG).

Solution 5

(b)

Mark scheme

- Since $\int_0^4 = \frac{1}{3} < \frac{1}{2}$, the median m satisfies $4 < m < 9$: $\frac{1}{3} + \frac{1}{3}\int_4^m x^{-1/2}\,dx = \frac{1}{2}$, i.e. $\frac{1}{3} + \frac{2}{3}(\sqrt{m} - 2) = \frac{1}{2}$. So $\sqrt{m} = \frac{9}{4}$, $m = \frac{81}{16} = 5.0625$.

Solution 5

(c)

Mark scheme

- CDF: $\mathrm{F}(x) = \frac{1}{24}x^{3/2}$ for $0 \leqslant x < 4$, $\frac{2}{3}\sqrt{x} - 1$ for $4 \leqslant x \leqslant 9$. With $Y = \sqrt{X}$, $\mathrm{G}(y) = \mathrm{F}(y^2) = \frac{1}{24}y^3$ for $0 \leqslant y < 2$, $\frac{2}{3}y - 1$ for $2 \leqslant y \leqslant 3$. Differentiating: $\mathrm{g}(y) = \frac{1}{8}y^2$ for $0 \leqslant y < 2$, $\frac{2}{3}$ for $2 \leqslant y \leqslant 3$, 0 otherwise.

Question 4

9231/42 N25 ▪ Q4 ▪ 10 marks

A continuous random variable X has cumulative distribution function F given by $F(x) = 0$ for $x < 1$, $F(x) = \frac{1}{5}x + a$ for $1 \leqslant x < 4$, $F(x) = \frac{1}{50}x^2 + b$ for $4 \leqslant x \leqslant 6$, and $F(x) = 1$ for $x > 6$, where a and b are constants.

(a) Find the value of a and the value of b. **[2]**

(b) Find the probability density function of X. **[2]**

(c) Given that $E(X) = \frac{529}{150}$, find $\mathrm{Var}(X)$. **[3]**

(d) Find the 10th and 90th percentiles of X. **[3]**

Solution 4

(a)

Mark scheme

- Continuity at $x = 1$: $\frac{1}{5} + a = 0 \Rightarrow a = -\frac{1}{5}$. At $x = 4$: $\frac{4}{5} - \frac{1}{5} = \frac{16}{50} + b \Rightarrow b = \frac{7}{25}$.

Solution 4

(b)

Mark scheme

- $f(x) = \frac{1}{5}$ for $1 \leqslant x < 4$; $f(x) = \frac{x}{25}$ for $4 \leqslant x \leqslant 6$; $f(x) = 0$ otherwise.

Solution 4

(c) Mark scheme

- $E(X^2) = \int_1^4 \frac{1}{5}x^2\,dx + \int_4^6 \frac{1}{25}x^3\,dx = \frac{21}{5} + \frac{52}{5} = \frac{73}{5}$.
  $\mathrm{Var}(X) = \frac{73}{5} - \left(\frac{529}{150}\right)^2 = \frac{48659}{22500} \approx 2.16$.

Solution 4

(d)

Mark scheme

- $F(x) = 0.1$ on $[1, 4)$: $\frac{1}{5}x - \frac{1}{5} = 0.1 \Rightarrow x = 1.5$. $F(x) = 0.9$ on $[4, 6]$: $\frac{1}{50}x^2 + \frac{7}{25} = 0.9 \Rightarrow x^2 = 31 \Rightarrow x = \sqrt{31}$. So $P_{10} = 1.5$, $P_{90} = \sqrt{31} \approx 5.57$.

Question 2

9231/44 N25 ▪ Q2 ▪ 10 marks

The continuous random variable X has probability density function f given by $\mathrm{f}(x) = \frac{4}{9}(x+1)$ for $0 \leqslant x \leqslant 1$, $(x-2)^2$ for $1 < x \leqslant 2$, and 0 otherwise.

(a) Find the cumulative distribution function of X. **[3]**

(b) Find the exact value of the median of X. **[2]**

(c) The random variable Y is defined by $Y = \sqrt{X}$. Find the cumulative distribution function of Y. **[3]**

(d) Determine whether the median of Y is greater than, or less than, the median of X. **[2]**

Solution 2

(a)

Mark scheme

- $\mathrm{F}(x) = 0$ for $x < 0$; $\frac{2}{9}x^2 + \frac{4}{9}x$ for $0 \leqslant x < 1$; $\frac{1}{3}(x-2)^3 + 1$ for $1 \leqslant x \leqslant 2$; 1 for $x > 2$.

Solution 2

(b)

Mark scheme

- Since $\mathrm{F}(1) = \frac{2}{3} > \frac{1}{2}$, the median lies in $0 \leqslant x < 1$: solve $\frac{2}{9}m^2 + \frac{4}{9}m = \frac{1}{2}$, giving $m = \frac{1}{2}\left(-2 + \sqrt{13}\right) \approx 0.803$.

Solution 2

(c)

Mark scheme

- With $Y = \sqrt{X}$, $\mathrm{G}(y) = \mathrm{F}(y^2) = 0$ for $y < 0$; $\frac{2}{9}y^4 + \frac{4}{9}y^2$ for $0 \leqslant y < 1$; $\frac{1}{3}(y^2 - 2)^3 + 1$ for $1 \leqslant y \leqslant \sqrt{2}$; 1 for $y > \sqrt{2}$.

Solution 2

(d)

Mark scheme

- $\mathrm{G}\left(\frac{1}{2}(-2+\sqrt{13})\right) = 0.379 < 0.5$, so the median of Y is greater than the median of X. (Equivalently $m_Y = \sqrt{m_X}$; since $m_X < 1$, $\sqrt{m_X} > m_X$, i.e. $m_Y = 0.896 > 0.803 = m_X$.)

Question 2

9231/41 J25 ▪ Q2 ▪ 12 marks

As shown in the diagram, the continuous random variable X has probability density function f given by $\mathrm{f}(x) = a$ for $0 \leqslant x \leqslant 5$, $b - cx$ for $5 \leqslant x \leqslant 8$, and 0 otherwise, where a, b and c are constants.

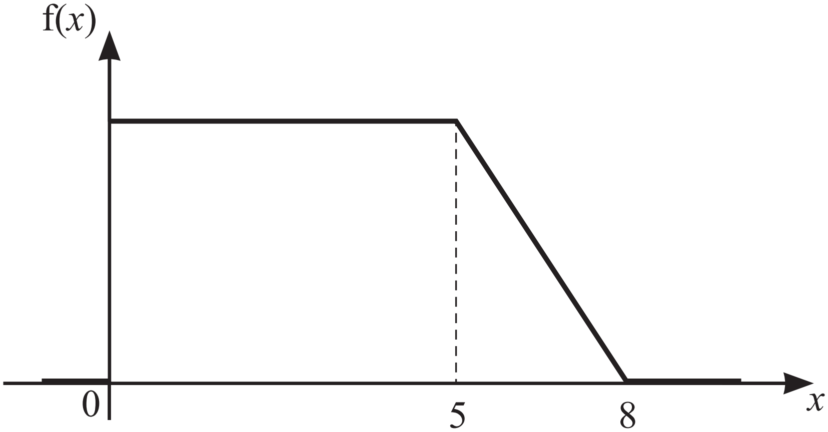
(a) Show that $a = \frac{2}{13}$ and find the values of b and c. **[3]**

(b) Find the mean of X. **[3]**

(c) Find the median of X. **[2]**

(d) The random variable Y is defined by $Y = X^2$. Find the cumulative distribution function for Y. **[4]**

Question 2



Solution 2

(a) Mark scheme

- Total area $= 1$: rectangle $5a$ plus triangle $\frac{1}{2} \cdot 3 \cdot a$, so $5a + \frac{3}{2}a = \frac{13}{2}a = 1$, giving $a = \frac{2}{13}$ (AG). The line $b - cx$ passes through $(5, a)$ and $(8, 0)$: $b = 8c$ and $a = 3c$, so $c = \frac{a}{3} = \frac{2}{39}$ and $b = \frac{16}{39}$.

Solution 2

(b) Mark scheme

- $\mathrm{E}(X) = \int_0^5 x \cdot \frac{2}{13}\, dx + \int_5^8 x\left(\frac{16}{39} - \frac{2}{39}x\right) dx = \left[\frac{1}{13}x^2\right]_0^5 + \left[\frac{8}{39}x^2 - \frac{2}{117}x^3\right]_5^8 = \frac{43}{13} = 3.31.$

Solution 2

(c)

Mark scheme

- The median m lies in $0 < m < 5$: $\frac{2}{13}m = 0.5$, so $m = \frac{13}{4} = 3.25$.

Solution 2

(d) Mark scheme

- CDF of X: $\mathrm{F}(x) = \frac{2}{13}x$ for $0 \leqslant x \leqslant 5$; $\frac{16}{39}x - \frac{1}{39}x^2 - \frac{25}{39}$ for $5 < x \leqslant 8$. With $Y = X^2$, $\mathrm{G}(y) = \mathrm{F}(\sqrt{y}) = 0$ for $y < 0$; $\frac{2}{13}\sqrt{y}$ for $0 \leqslant y \leqslant 25$; $\frac{16}{39}\sqrt{y} - \frac{1}{39}y - \frac{25}{39}$ for $25 < y \leqslant 64$; 1 for $y > 64$.

Question 3

9231/43 J25 ▪ Q3 ▪ 10 marks

A continuous random variable X has probability density function f given by $\mathrm{f}(x) = kx$ for $0 \leqslant x < 1$, $k(8 - x)$ for $1 \leqslant x \leqslant 8$, and 0 otherwise, where k is a constant.

(a) Show that $k = \frac{1}{25}$. **[2]**

(b) Find the median value of X. **[3]**

(c) The random variable Y is defined by $Y = \sqrt[3]{X}$. Find the probability density function of Y. **[5]**

Solution 3

(a)

Mark scheme

- $\displaystyle\int_0^1 kx\,dx + \int_1^8 k(8-x)\,dx = 1$: $\left[\frac{1}{2}kx^2\right]_0^1 + \left[8kx - \frac{1}{2}kx^2\right]_1^8 = \frac{1}{2}k + (64k - 32k) - \left(8k - \frac{1}{2}k\right) = 25k = 1$, so $k = \frac{1}{25}$ (AG).

Solution 3

(b)

Mark scheme

- The median m lies in $1 \leqslant m \leqslant 8$: $\int_0^1 kx\,dx + \int_1^m k(8-x)\,dx = \frac{1}{2}$ gives $m^2 - 16m + 39 = 0$, so $m = 3$ (rejecting $m = 13$).

Solution 3

(c) Mark scheme

- CDF of X: $\mathrm{F}(x) = \frac{1}{50}x^2$ for $0 \leqslant x < 1$; $\frac{8}{25}x - \frac{1}{50}x^2 - \frac{7}{25}$ for $1 \leqslant x \leqslant 8$. With $Y = \sqrt[3]{X}$, $\mathrm{G}(y) = \mathrm{F}(y^3) = \frac{1}{50}y^6$ for $0 \leqslant y < 1$; $\frac{8}{25}y^3 - \frac{1}{50}y^6 - \frac{7}{25}$ for $1 \leqslant y \leqslant 2$. Differentiating, $\mathrm{g}(y) = \frac{3}{25}y^5$ for $0 \leqslant y < 1$; $\frac{24}{25}y^2 - \frac{3}{25}y^5$ for $1 \leqslant y \leqslant 2$; 0 otherwise.