Past-paper walk-through

Equilibrium of a rigid body ▪ 3.2

eXercise

Questions in this set

- 9231/31 N25 Q5 ▪ 9 marks
- 9231/32 N25 Q3 ▪ 6 marks
- 9231/34 N25 Q3 ▪ 7 marks
- 9231/31 J25 Q4 ▪ 7 marks
- 9231/33 J25 Q4 ▪ 4 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 5

9231/31 N25 ▪ Q5 ▪ 9 marks

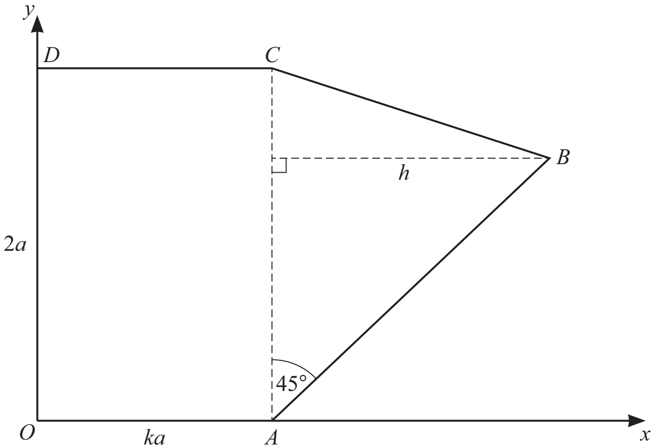
A uniform lamina *OABCD* consists of a rectangle *OACD* and a triangle *ABC*. The length of *OA* is ka, the length of *OD* is $2a$, the height of triangle *ABC* is h and angle *CAB* is 45 (see diagram). Relative to axes through *O*, parallel and perpendicular to *OA* as shown, the centre of mass of triangle *ABC* is $(\bar{x}, \bar{y})$.

(a) Show that $\bar{x}$ is $\frac{1}{3}(3ka + h)$, and find an expression for $\bar{y}$. **[3]**

(b) The lamina *OABCD* is placed vertically on its edge *OA* on a horizontal plane. Find, in terms of a and k, the set of values of h for which the lamina is in equilibrium. **[4]**

(c) It is now given that $k = \frac{\sqrt{3}}{3}$ and that the lamina is on the point of toppling. Find, in terms of a, the coordinates of the centre of mass of the triangle *ABC*. **[2]**

Question 5



Solution 5

(a)

Mark scheme

- x-coordinates of A, B, C are ka, $ka + h$, ka, so $\bar{x} = \frac{1}{3}(ka + (ka + h) + ka) = \frac{1}{3}(3ka + h)$ (AG). y-coordinates are 0, h, $2a$, so $\bar{y} = \frac{1}{3}(0 + h + 2a) = \frac{1}{3}(2a + h)$.

Solution 5

(b)

Mark scheme

- Taking moments about O (the y-axis) for the whole lamina: $\bar{X}(2ka^2 + ah) = 2ka^2 \cdot \frac{1}{2}ka + ah\left(ka + \frac{1}{3}h\right)$. For equilibrium the centre of mass must lie over the base OA, i.e. $\bar{X} \leqslant ka$: $\dfrac{a^2k^2 + ahk + \frac{1}{3}h^2}{2ak + h} \leqslant ka$. Simplifying gives $0 \leqslant h \leqslant \sqrt{3}\,ka$.

Solution 5

(c)

Mark scheme

- On the point of toppling, $h = \sqrt{3}\,ka$; with $k = \frac{\sqrt{3}}{3}$ this gives $h = a$. Then $\bar{x} = \frac{1}{3}(3ka + h) = \frac{1}{3}a(\sqrt{3} + 1)$ and (since $h = a$, the triangle is isosceles) $\bar{y} = a$. Centre of mass of ABC: $\left(\frac{1}{3}a(\sqrt{3} + 1),\ a\right)$.

Question 3

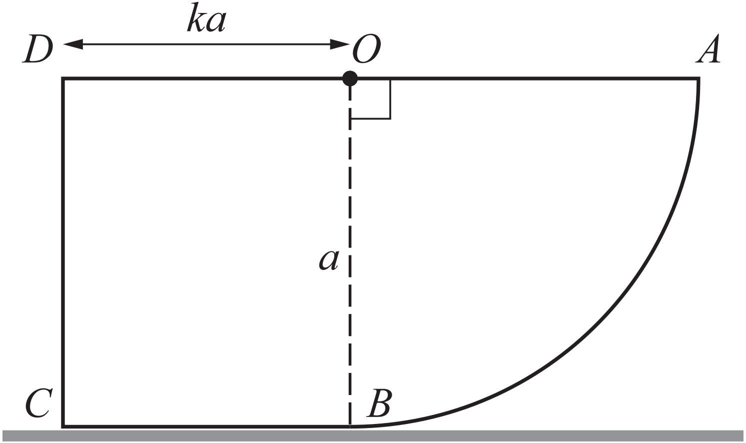
9231/32 N25 ▪ Q3 ▪ 6 marks

A uniform lamina *OABCD* is in the form of a rectangle *OBCD* joined along the edge *OB* to a quarter circle *OAB*. The length of *DO* is ka and the length of *OB* is a. The lamina rests in a vertical plane with its edge *CB* on a horizontal surface (see diagram).

(a) Find, in terms of k, a and π, an expression for the distance of the centre of mass above the horizontal surface. [You may use without proof the result for the centre of mass of a circular sector in the list of formulae (MF19).] **[4]**

(b) The lamina is on the point of toppling about B. Find the value of k. **[2]**

Question 3



Solution 3

(a) Mark scheme

- Rectangle: area ka^2, height $\frac{a}{2}$. Quarter circle (centre O, radius a): area $\frac{\pi a^2}{4}$, height $a - \frac{4a}{3\pi}$. $\bar{y} = \dfrac{ka^2 \cdot \frac{a}{2} + \frac{\pi a^2}{4}\left(a - \frac{4a}{3\pi}\right)}{ka^2 + \frac{\pi a^2}{4}} = \dfrac{a(6k + 3\pi - 4)}{3(4k + \pi)}$.

Solution 3

(b)

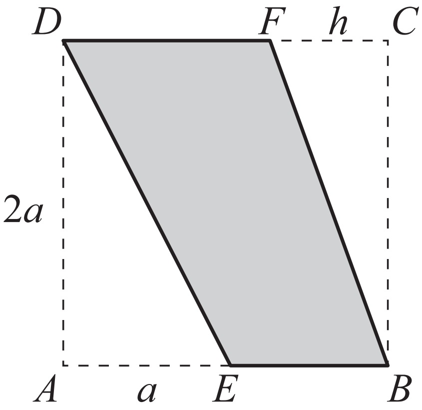
Mark scheme

- Toppling about B requires the centre of mass directly above B, i.e. $\bar{x} = 0$:
  $-\frac{k^2 a^3}{2} + \frac{a^3}{3} = 0 \Rightarrow k^2 = \frac{2}{3} \Rightarrow k = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$.

Question 3

9231/34 N25 ▪ Q3 ▪ 7 marks

The lamina $BFDE$ is obtained by removing triangles AED and BCF from a uniform square lamina $ABCD$ of side $2a$. The length of side AE is a and the length of side FC is h (see diagram). The centre of mass of $BFDE$ is at a distance $\bar{x}$ from AD, and at a distance $\bar{y}$ from AB.



(a) Show that $\bar{x} = \dfrac{h^2 - 6ah + 11a^2}{3(3a - h)}$ and find a corresponding expression for $\bar{y}$. **[5]**

(b) The lamina $BFDE$ is placed vertically on its edge EB on a smooth horizontal surface. Find, in terms of a, the set of possible values of h for which the lamina

Solution 3

(a)

Mark scheme

- Using areas and distances ($ABCD$: $4a^2$, a, a; AED: a^2, $\frac{1}{3}a$, $\frac{2}{3}a$; BCF: ah, $2a - \frac{1}{3}h$, $\frac{4}{3}a$; shape $BFDE$: $3a^2 - ah$). Moments about AD: $(3a^2 - ah)\bar{x} = 4a^2 \cdot a - a^2 \cdot \frac{1}{3}a - ah\left(2a - \frac{1}{3}h\right)$, giving $\bar{x} = \dfrac{h^2 - 6ah + 11a^2}{3(3a - h)}$ (AG). Moments about AB: $(3a^2 - ah)\bar{y} = 4a^2 \cdot a - a^2 \cdot \frac{2}{3}a - ah \cdot \frac{4}{3}a$, giving $\bar{y} = \dfrac{2a(5a - 2h)}{3(3a - h)}$.

Solution 3

(b)

Mark scheme

- For equilibrium on edge EB, the centre of mass must lie over the base, i.e. $\bar{x} \geqslant a$: $h^2 - 3ah + 2a^2 \geqslant 0$, i.e. $(h-a)(h-2a) \geqslant 0$. So $0 \leqslant h \leqslant a$ (or $h = 2a$).

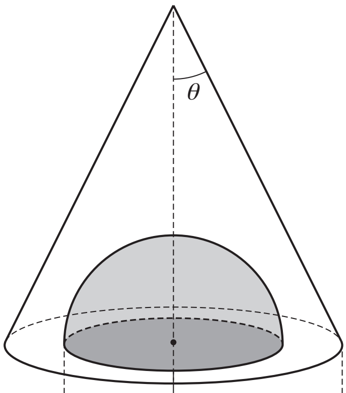
Question 4

9231/31 J25 ▪ Q4 ▪ 7 marks

An object is formed by removing a solid hemisphere, radius $2r$, from a uniform solid cone, radius $3r$ and semi-vertical angle θ, where $\tan\theta = \frac{1}{2}$. The axes of symmetry of the cone and the hemisphere coincide. The base of the cone and the base of the hemisphere are in the same plane as each other (see diagram).

(a) Find, in terms of r, the distance of the centre of mass of the object from its base. **[4]**

(b) The object is placed such that its circular base makes contact with a rough



Solution 4

(a) Mark scheme

- Cone: height $= 6r$ (since $\tan\theta = \frac{3r}{H} = \frac{1}{2}$), volume $18\pi r^3$, CoM $\frac{3}{2}r$ above base. Hemisphere: volume $\frac{16}{3}\pi r^3$, CoM $\frac{3}{4}r$ above base. Object: volume $18\pi r^3 - \frac{16}{3}\pi r^3 = \frac{38}{3}\pi r^3$. Moments: $\left(\frac{38}{3}\pi r^3\right)\bar{y} = 18\pi r^3\left(\frac{3}{2}r\right) - \frac{16}{3}\pi r^3\left(\frac{3}{4}r\right)$, giving $\bar{y} = \dfrac{69}{38}r$.

Solution 4

(b)

Mark scheme

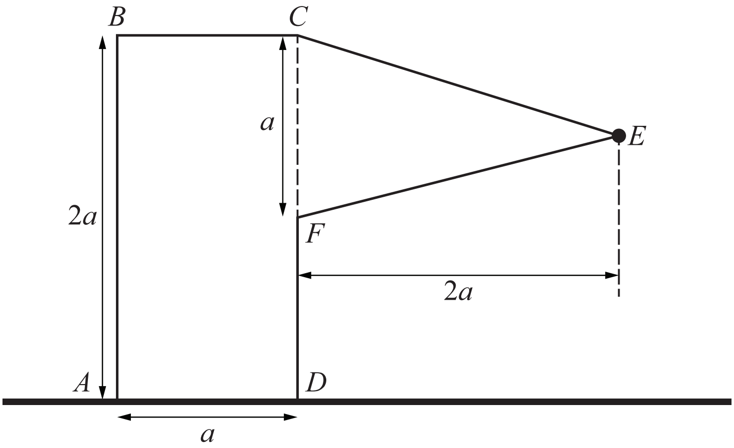
- On the point of toppling, $\tan\alpha = \dfrac{3r}{\bar{y}} = \dfrac{3r}{\frac{69}{38}r} = \dfrac{38}{23}$, so $\alpha = 58.8$.

Question 4

9231/33 J25 ▪ Q4 ▪ 4 marks

An object consists of a uniform lamina with a particle attached. The uniform lamina $ABCEFD$ of mass m is formed from a rectangle $ABCD$ and an isosceles triangle CEF, where F is the midpoint of CD. The rectangle has sides $AB = 2a$ and $AD = a$. The triangle CEF has base a and height $2a$. The particle of mass km is attached to the lamina at E. The object rests in a vertical plane with its edge AD on horizontal ground (see diagram). Given that the object is on the point of toppling in its vertical plane about the vertex D, find the value of k. **[4]**

Question 4



Solution 4

Mark scheme

- Lamina (mass m): rectangle area $2a^2$, distance $\frac{1}{2}a$ to the left of D; triangle area a^2, distance $\frac{2}{3}a$ to the right of D. Moments give the lamina's centre of mass $\frac{1}{9}a$ to the left of D. On the point of toppling about D, the particle's moment balances the lamina's: $m\left(\frac{1}{9}a\right) = km(2a)$, so $k = \frac{1}{18}$.