Past-paper walk-through

Differential equations ▪ 2.6

eXercise

Questions in this set

- 9231/21 N25 Q4 ▪ 10 marks
- 9231/21 N25 Q7 ▪ 12 marks
- 9231/22 N25 Q5 ▪ 10 marks
- 9231/24 N25 Q5 ▪ 10 marks
- 9231/24 N25 Q6 ▪ 10 marks
- 9231/21 J25 Q5 ▪ 10 marks
- 9231/21 J25 Q7 ▪ 10 marks
- 9231/23 J25 Q4 ▪ 10 marks
- 9231/23 J25 Q7 ▪ 9 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

9231/21 N25 ▪ Q4 ▪ 10 marks

Find the particular solution of the differential equation $5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 5x + 3$, given that, when $x = 0$, $y = \frac{dy}{dx} = 0$. **[10]**

Solution 4

Mark scheme

- Auxiliary equation $5m^2 + 2m + 1 = 0 \Rightarrow m = -\frac{1}{5} \pm \frac{2}{5}\mathrm{i}$, so CF $= \mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x + B\sin\frac{2}{5}x\right)$. PI $y = px^2 + qx + r$ gives $p = 1,\ q = 1,\ r = -9$. General solution $y = \mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x + B\sin\frac{2}{5}x\right) + x^2 + x - 9$. Initial conditions: $A - 9 = 0$ and $\frac{2}{5}B - \frac{1}{5}A + 1 = 0 \Rightarrow A = 9,\ B = 2$. So $y = \mathrm{e}^{-\frac{1}{5}x}\left(9\cos\frac{2}{5}x + 2\sin\frac{2}{5}x\right) + x^2 + x - 9$.

Question 7

9231/21 N25 ▪ Q7 ▪ 12 marks

(a) Show that $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$. **[3]**

(b) Find the solution of the differential equation $x\dfrac{dy}{dx} - y = x^2 \tanh^{-1} x$, for $0 < x < 1$, given that $y = 0$ when $x = \frac{1}{2}$. Give your answer in an exact form. **[9]**

Solution 7

(a) Mark scheme

- Using $\tanh^{-1} x = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$:
  $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{2}\cdot\dfrac{1-x}{1+x}\cdot\dfrac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} = \frac{1}{2}\cdot\dfrac{1-x}{1+x}\cdot\dfrac{2}{(1-x)^2} = \dfrac{1}{1-x^2}$
  (AG). (Alternatively $\tanh y = x \Rightarrow \operatorname{sech}^2 y\,\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$.)

Solution 7

(b) Mark scheme

- $\dfrac{dy}{dx} - \dfrac{y}{x} = x\tanh^{-1}x$; integrating factor $\mathrm{e}^{-\int x^{-1}dx} = x^{-1}$, so $\dfrac{d}{dx}(x^{-1}y) = \tanh^{-1}x$. Then $x^{-1}y = x\tanh^{-1}x - \displaystyle\int \frac{x}{1-x^2}\,dx = x\tanh^{-1}x + \tfrac{1}{2}\ln(1-x^2) + C$. Using $y = 0$ at $x = \tfrac{1}{2}$: $C = -\tfrac{1}{4}\ln\tfrac{27}{16}$. So $x^{-1}y = x\tanh^{-1}x + \tfrac{1}{2}\ln(1-x^2) - \tfrac{1}{4}\ln\tfrac{27}{16}$ (i.e. $y = x\left(x\tanh^{-1}x + \tfrac{1}{2}\ln(1-x^2) - \tfrac{1}{4}\ln\tfrac{27}{16}\right)$).

Question 5

9231/22 N25 ▪ Q5 ▪ 10 marks

(a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 5\cos x$. **[7]**

(b) For large positive values of x and for any initial conditions, show that the solution to part (a) can be approximated by $y \approx R\sin(x + \phi)$, where the constants R and ϕ are to be determined. **[3]**

Solution 5

(a)

Mark scheme

- Auxiliary equation $m^2 + 4m + 3 = 0 \Rightarrow m = -1, -3$; CF $= Ae^{-x} + Be^{-3x}$. Particular integral $y = \frac{1}{2}\cos x + \sin x$. General solution $y = Ae^{-x} + Be^{-3x} + \frac{1}{2}\cos x + \sin x$.

Solution 5

(b)

Mark scheme

- As $x \to \infty$, $Ae^{-x}, Be^{-3x} \to 0$, so $y \approx \frac{1}{2}\cos x + \sin x = R\sin(x+\phi)$ with $R = \frac{\sqrt{5}}{2}$ and $\phi = \tan^{-1}\frac{1}{2} = 0.4636$.

Question 5

9231/24 N25 ▪ Q5 ▪ 10 marks

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x + \cos 2x.$$ **[7]**

(b) For large positive values of x and for any initial conditions, show that the solution to part (a) can be approximated by $y \approx R\sin(2x - \phi)$, where R and ϕ are positive constants to be determined. **[3]**

Solution 5

(a) Mark scheme

- Auxiliary equation $m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,\mathrm{i}$, so CF $= \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$. PI $y = p\sin 2x + q\cos 2x$: substituting gives $-3p - 2q = 1$ and $2p - 3q = 1$, so $p = -\frac{1}{13}$, $q = -\frac{5}{13}$. General solution $y = \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}\sin 2x - \frac{5}{13}\cos 2x$.

Solution 5

(b) Mark scheme

- For large x, $\mathrm{e}^{-\frac{1}{2}x} \to 0$, so $y \approx -\frac{1}{13}\sin 2x - \frac{5}{13}\cos 2x = \sqrt{\frac{2}{13}}\sin(2x - 1.77)$ (i.e. $R = \sqrt{\frac{2}{13}} \approx 0.392$, $\phi = 1.77$ rad ≈ 101.3).

Question 6

9231/24 N25 ▪ Q6 ▪ 10 marks

Find the particular solution of the differential equation $\dfrac{dy}{dx} + \dfrac{1+x}{1-2x-x^2}y = 1$, given that $y = \pi$ when $x = 0$. Give your answer in an exact form. **[10]**

Solution 6

- $\displaystyle\int \frac{1+x}{1-2x-x^2}\,dx = -\tfrac{1}{2}\ln(1-2x-x^2)$, so the integrating factor is $e^{-\frac{1}{2}\ln(1-2x-x^2)} = \dfrac{1}{\sqrt{1-2x-x^2}}$. Then $\dfrac{d}{dx}\left(\dfrac{y}{\sqrt{1-2x-x^2}}\right) = \dfrac{1}{\sqrt{1-2x-x^2}}$.
  Completing the square $1-2x-x^2 = 2-(x+1)^2$:
  $\dfrac{y}{\sqrt{1-2x-x^2}} = \sin^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right) + C$. Using $y=\pi$ at $x=0$:
  $\pi = \frac{1}{4}\pi + C \Rightarrow C = \frac{3}{4}\pi$. So $\dfrac{y}{\sqrt{1-2x-x^2}} = \sin^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right) + \frac{3}{4}\pi$.

Question 5

9231/21 J25 ▪ Q5 ▪ 10 marks

Find the particular solution of the differential equation $6\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 6x = e^{-t}$, given that, when $t = 0$, $x = \dfrac{dx}{dt} = 0$. **[10]**

Solution 5

- Auxiliary equation $6m^2 + 3m + 6 = 0 \Rightarrow m = -\frac{1}{4} \pm \frac{1}{4}\mathrm{i}\sqrt{15}$, so CF $= \mathrm{e}^{-\frac{1}{4}t}\left(A\cos\frac{\sqrt{15}}{4}t + B\sin\frac{\sqrt{15}}{4}t\right)$. PI $x = k\mathrm{e}^{-t}$: $6k - 3k + 6k = 1 \Rightarrow k = \frac{1}{9}$. General solution $x = \mathrm{e}^{-\frac{1}{4}t}\left(A\cos\frac{\sqrt{15}}{4}t + B\sin\frac{\sqrt{15}}{4}t\right) + \frac{1}{9}\mathrm{e}^{-t}$. Initial conditions: $A = -\frac{1}{9}$, $B = \frac{1}{3\sqrt{15}}$. So $x = \frac{1}{3}\mathrm{e}^{-\frac{1}{4}t}\left(-\frac{1}{3}\cos\frac{\sqrt{15}}{4}t + \frac{1}{\sqrt{15}}\sin\frac{\sqrt{15}}{4}t\right) + \frac{1}{9}\mathrm{e}^{-t}$.

Question 7

9231/21 J25 ▪ Q7 ▪ 10 marks

Find the solution of the differential equation $\dfrac{dy}{dx} - \dfrac{x+5}{x^2+10x+61}y = 1$, given that $y = 0$ when $x = 3$. Give your answer in an exact form. **[10]**

Solution 7

- $\displaystyle\int \frac{x+5}{x^2+10x+61}\,dx = \tfrac{1}{2}\ln(x^2+10x+61)$, so the integrating factor is $\displaystyle e^{-\frac{1}{2}\ln(x^2+10x+61)} = \frac{1}{\sqrt{x^2+10x+61}}$. Then $\displaystyle\frac{d}{dx}\left(\frac{y}{\sqrt{x^2+10x+61}}\right) = \frac{1}{\sqrt{x^2+10x+61}}$. Completing the square $x^2+10x+61 = (x+5)^2+36$: $\displaystyle\frac{y}{\sqrt{x^2+10x+61}} = \sinh^{-1}\left(\frac{x+5}{6}\right) + C$. Using $y=0$ at $x=3$: $C = -\sinh^{-1}\frac{4}{3} = -\ln 3$. So $\displaystyle\frac{y}{\sqrt{x^2+10x+61}} = \sinh^{-1}\left(\frac{x+5}{6}\right) - \sinh^{-1}\frac{4}{3}$.

Solution 7

Mark scheme

Question 4

9231/23 J25 ▪ Q4 ▪ 10 marks

Find the particular solution of the differential equation $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 2t^2 + t - 1$, given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. **[10]**

Solution 4

Mark scheme

- Auxiliary equation $m^2 + m - 2 = 0 \Rightarrow m = 1, -2$, so CF $= Ae^t + Be^{-2t}$. PI $x = pt^2 + qt + r$ gives $p = -1$, $q = -\frac{3}{2}$, $r = -\frac{5}{4}$. General solution $x = Ae^t + Be^{-2t} - t^2 - \frac{3}{2}t - \frac{5}{4}$. Initial conditions: $A + B - \frac{5}{4} = 0$ and $A - 2B - \frac{3}{2} = 0 \Rightarrow A = \frac{4}{3}$, $B = -\frac{1}{12}$. So $x = \frac{4}{3}e^t - \frac{1}{12}e^{-2t} - t^2 - \frac{3}{2}t - \frac{5}{4}$.

Question 7

9231/23 J25 ▪ Q7 ▪ 9 marks

Find the solution of the differential equation $\dfrac{dy}{dx} - \dfrac{2x+6}{x^2+6x+5}y = 4$, given that $y = 0$ when $x = 0$. Give your answer in an exact form. **[9]**

Solution 7

- $\displaystyle\int \frac{2x+6}{x^2+6x+5}\,dx = \ln(x^2+6x+5)$, so the integrating factor is $\dfrac{1}{x^2+6x+5}$. Then $\dfrac{d}{dx}\left(\dfrac{y}{x^2+6x+5}\right) = \dfrac{4}{x^2+6x+5}$. Partial fractions: $\dfrac{4}{x^2+6x+5} = \dfrac{1}{x+1} - \dfrac{1}{x+5}$, so $\dfrac{y}{x^2+6x+5} = \ln\left(\dfrac{x+1}{x+5}\right) + C$. Using $y=0$ at $x=0$: $C = -\ln\frac{1}{5}$. So $\dfrac{y}{x^2+6x+5} = \ln\left(\dfrac{x+1}{x+5}\right) - \ln\frac{1}{5} = \ln\left(\dfrac{5x+5}{x+5}\right)$.