Past-paper walk-through

Complex numbers ▪ 2.5

eXercise

Questions in this set

- 9231/21 N25 Q8 ▪ 12 marks
- 9231/22 N25 Q2 ▪ 5 marks
- 9231/22 N25 Q4 ▪ 8 marks
- 9231/24 N25 Q3 ▪ 8 marks
- 9231/21 J25 Q1 ▪ 5 marks
- 9231/21 J25 Q3 ▪ 6 marks
- 9231/23 J25 Q5 ▪ 10 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 8

9231/21 N25 ▪ Q8 ▪ 12 marks

(a) State the sum of the series $z + z^2 + \ldots + z^n$, for $z \neq 1$. **[1]**

(b) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$ use de Moivre's theorem to deduce that

$$\sum_{m=1}^{n} \left(\tfrac{1}{2}\right)^m \sin m\theta = \frac{\left(\frac{1}{2}\right)^{n+2}\sin n\theta - \left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta + \frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta}.$$

[6]

(c) Use the result in (b) to find $\sum_{m=1}^{n} \left(\tfrac{1}{2}\right)^m m\cos m\theta$ in terms of n and θ. **[3]**

(d) Hence find $\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m m\cos m\theta$ in terms of $\cos\theta$. **[2]**

Solution 8

(a)

Mark scheme

- $z + z^2 + \ldots + z^n = \dfrac{z^{n+1} - z}{z - 1}.$

Solution 8

(b)

Mark scheme

- With $z = \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$, $\sum_{m=1}^{n} \left(\frac{1}{2}\right)^m \mathrm{e}^{\mathrm{i}m\theta} = \dfrac{z^{n+1} - z}{z - 1}$. Multiplying numerator and denominator by the conjugate of $z - 1$ and taking the imaginary part (using $\sin(A - B)$) gives $\sum_{m=1}^{n} \left(\frac{1}{2}\right)^m \sin m\theta = \dfrac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta}$ (AG).

Solution 8

(c)

Mark scheme

- $\sum_{m=1}^{n} \left(\frac{1}{2}\right)^m m \cos m\theta = \frac{d}{d\theta}\left[\sum_{m=1}^{n} \left(\frac{1}{2}\right)^m \sin m\theta\right]$. Differentiating the result in (b) by the quotient rule: $=$

$$\frac{\left(\frac{5}{4} - \cos\theta\right)\left(\left(\frac{1}{2}\right)^{n+2} n\cos n\theta - \left(\frac{1}{2}\right)^{n+1}(n+1)\cos(n+1)\theta + \frac{1}{2}\cos\theta\right) - \left(\left(\frac{1}{2}\right)^{n+2} \sin n\theta\right.}{\left(\frac{5}{4} - \cos\theta\right)^2}$$

Solution 8

(d)

Mark scheme

- As $n \to \infty$ the $\left(\frac{1}{2}\right)^{n+\cdots}$ terms vanish, so
$$\sum_{m=1}^{\infty} \left(\tfrac{1}{2}\right)^m m \cos m\theta = \frac{\left(\frac{5}{4} - \cos\theta\right)\left(\frac{1}{2}\cos\theta\right) - \left(\frac{1}{2}\sin\theta\right)\sin\theta}{\left(\frac{5}{4} - \cos\theta\right)^2} = \frac{\frac{5}{8}\cos\theta - \frac{1}{2}}{\left(\frac{5}{4} - \cos\theta\right)^2}.$$

Question 2

9231/22 N25 ▪ Q2 ▪ 5 marks

Find the roots of the equation $z^4 = 8 - 8i\sqrt{3}$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. **[5]**

Solution 2

Mark scheme

- $8 - 8i\sqrt{3} = 16\, e^{-i\pi/3}$. Fourth roots have $r = 2$ and $\theta = -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, -\frac{7\pi}{12}$: $2e^{-i\pi/12}$, $2e^{i5\pi/12}$, $2e^{i11\pi/12}$, $2e^{-i7\pi/12}$.

Question 4

9231/22 N25 ▪ Q4 ▪ 8 marks

(a) By considering the binomial expansion of $\left(z + \dfrac{1}{z}\right)^6$, where $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that $\cos^6\theta = a\cos 6\theta + b\cos 4\theta + c\cos 2\theta + d$, where a, b, c and d are constants to be determined. **[5]**

(b) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \cos^6 2x\,dx$. **[3]**

Solution 4

(a)

Mark scheme

- $z + \frac{1}{z} = 2\cos\theta$ and $(z + \frac{1}{z})^6 = (z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$. So $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$, giving $a = \frac{1}{32}$, $b = \frac{3}{16}$, $c = \frac{15}{32}$, $d = \frac{5}{16}$.

Solution 4

(b)

Mark scheme

- With $\theta = 2x$, integrating over $[0, \frac{\pi}{2}]$ all the cosine terms vanish, leaving $\displaystyle\int_0^{\pi/2} \cos^6 2x\,dx = \frac{5}{16} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$.

Question 3

9231/24 N25 ▪ Q3 ▪ 8 marks

(a) Use de Moivre's theorem to show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. **[4]**

(b) Hence obtain the roots of the equation $32x^5 - 40x^3 + 10x - \sqrt{3} = 0$ in the form $\sin q\pi$, where q is a rational number. **[4]**

Solution 3

(a)

Mark scheme

- $\sin 5\theta = \operatorname{Im}\left((\cos\theta + \mathrm{i}\sin\theta)^5\right) = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$.
  Using $\cos^2\theta = 1 - \sin^2\theta$:
  $= \sin^5\theta - 10\sin^3\theta(1-\sin^2\theta) + 5\sin\theta(1-\sin^2\theta)^2 = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$
  (AG).

Solution 3

(b) Mark scheme

- Let $x = \sin\theta$; then $32x^5 - 40x^3 + 10x = 2\sin 5\theta = \sqrt{3}$, so $\sin 5\theta = \frac{1}{2}\sqrt{3}$. Thus $5\theta = \frac{1}{3}\pi + 2k\pi$ or $\frac{2}{3}\pi + 2k\pi$, giving roots $\sin\frac{1}{15}\pi$, $\sin\frac{2}{15}\pi$, $\sin\left(-\frac{5}{15}\pi\right)$, $\sin\left(-\frac{4}{15}\pi\right)$, $\sin\frac{7}{15}\pi$ (i.e. $q = \frac{1}{15}, \frac{2}{15}, -\frac{1}{3}, -\frac{4}{15}, \frac{7}{15}$).

Question 1

9231/21 J25 ▪ Q1 ▪ 5 marks

Find the roots of the equation $z^3 = 27 - 27\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. **[5]**

Solution 1

Mark scheme

- $z^3 = 27 - 27\mathrm{i} = 27\sqrt{2}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}$. Cube roots have modulus $(27\sqrt{2})^{1/3} = 3 \cdot 2^{1/6}$ and arguments $-\frac{1}{12}\pi + \frac{2}{3}k\pi$: $z_1 = 3 \cdot 2^{1/6}\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}}$, $z_2 = 3 \cdot 2^{1/6}\mathrm{e}^{\frac{7}{12}\pi\mathrm{i}}$, $z_3 = 3 \cdot 2^{1/6}\mathrm{e}^{-\frac{9}{12}\pi\mathrm{i}}$.

Question 3

9231/21 J25 ▪ Q3 ▪ 6 marks

By considering the binomial expansion of $\left(z - \frac{1}{z}\right)^5$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that $\operatorname{cosec}^5\theta = \frac{a}{\sin 5\theta + b\sin 3\theta + c\sin\theta}$, where a, b and c are integers to be determined. **[6]**

Solution 3

Mark scheme

- With $z - z^{-1} = 2\mathrm{i}\sin\theta$: $(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$.
  Using $z^n - z^{-n} = 2\mathrm{i}\sin n\theta$: $2^5\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$,
  so $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$, giving
  $\mathrm{cosec}^5\theta = \dfrac{16}{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}$ $(a = 16,\ b = -5,\ c = 10)$.

Question 5

9231/23 J25 ▪ Q5 ▪ 10 marks

(a) Use de Moivre's theorem to show that $\sec 5\theta = \dfrac{\sec^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16}$. **[6]**

(b) Hence, obtain the roots of the equation $\sqrt{3}\,x^5 - 10x^4 + 40x^2 - 32 = 0$ in the form $\sec(q\pi)$, where q is rational. **[4]**

Solution 5

(a)

Mark scheme

- $(\cos 5\theta + \mathrm{i}\sin 5\theta) = (c + \mathrm{i}s)^5$ ($c = \cos\theta$, $s = \sin\theta$). Real part: $\cos 5\theta = c^5 - 10s^2c^3 + 5s^4c$. Using $s^2 = 1 - c^2$: $= 16c^5 - 20c^3 + 5c$. Dividing numerator and denominator of $\sec 5\theta = \dfrac{1}{16c^5 - 20c^3 + 5c}$ by c^5: $\sec 5\theta = \dfrac{\sec^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16}$ (AG).

Solution 5

(b) Mark scheme

- $\sqrt{3}\,x^5 - 10x^4 + 40x^2 - 32 = 0 \Rightarrow x^5 = \frac{2}{\sqrt{3}}(5x^4 - 20x^2 + 16)$, so $\dfrac{x^5}{5x^4 - 20x^2 + 16} = \frac{2}{\sqrt{3}}$. With $x = \sec\theta$, $\sec 5\theta = \frac{2}{\sqrt{3}}$, i.e. $\cos 5\theta = \frac{\sqrt{3}}{2}$. Roots: $\sec\frac{1}{30}\pi$, $\sec\frac{11}{30}\pi$, $\sec\frac{13}{30}\pi$, $\sec\frac{23}{30}\pi$, $\sec\frac{25}{30}\pi$ $(\approx -1.35, -1.15, 1.01, 2.46, 4.81)$.