Past-paper walk-through

Integration ▪ 2.4

eXercise

Questions in this set

- 9231/21 N25 Q3 ▪ 9 marks
- 9231/21 N25 Q5 ▪ 11 marks
- 9231/22 N25 Q1 ▪ 4 marks
- 9231/22 N25 Q6 ▪ 13 marks
- 9231/24 N25 Q1 ▪ 5 marks
- 9231/24 N25 Q4 ▪ 10 marks
- 9231/21 J25 Q2 ▪ 7 marks
- 9231/21 J25 Q4 ▪ 9 marks
- 9231/23 J25 Q6 ▪ 10 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

9231/21 N25 ▪ Q3 ▪ 9 marks

The integral I_n is defined by $I_n = \int_0^1 (1+x^2)^n \, dx$.

(a) By considering $\frac{d}{dx}\left(x(1+x^2)^n\right)$, or otherwise, show that $(2n+1)I_n = 2^n + 2nI_{n-1}$. **[5]**

(b) Find the exact value of I_{-2}. **[4]**

Solution 3

(a)

Mark scheme

- $\dfrac{d}{dx}\left(x(1+x^2)^n\right) = 2nx^2(1+x^2)^{n-1} + (1+x^2)^n$. Using $x^2 = (1+x^2) - 1$: $= 2n(1+x^2)^n - 2n(1+x^2)^{n-1} + (1+x^2)^n$. Integrating from 0 to 1: $\left[x(1+x^2)^n\right]_0^1 = 2nI_n - 2nI_{n-1} + I_n$, i.e. $2^n = (2n+1)I_n - 2nI_{n-1}$, so $(2n+1)I_n = 2^n + 2nI_{n-1}$ (AG).

Solution 3

(b)

Mark scheme

- $I_{-1} = \left[\tan^{-1} x\right]_0^1 = \frac{1}{4}\pi$. Applying the reduction formula with $n = -1$: $-I_{-1} = \frac{1}{2} - 2I_{-2}$, so $-\frac{1}{4}\pi = \frac{1}{2} - 2I_{-2}$, giving $I_{-2} = \frac{1}{4} + \frac{1}{8}\pi$.

Question 5

9231/21 N25 ▪ Q5 ▪ 11 marks

The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

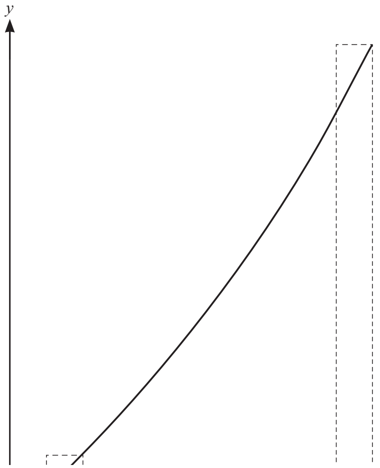
$\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx < U_n$, where $U_n = \frac{1}{12}\left(1 + \frac{1}{n}\right)\left(7 + \frac{1}{n}\right)$. **[5]**

(b) Use a similar method to find, in terms of n, a lower bound L_n for

$\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx$. **[4]**

(c) Show that $\lim_{n\to\infty}(U_n - L_n) = 0$. **[2]**

Question 5



Question 5

Solution 5

(a) Mark scheme

- Right-endpoint rectangle sum $= \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{3}\left(\frac{k}{n}\right)^3 + \frac{k}{n}\right) = \frac{1}{3n^4} \cdot \frac{n^2(n+1)^2}{4} + \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{12}\left(1+\frac{1}{n}\right)^2 + \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{12}\left(1+\frac{1}{n}\right)\left(7+\frac{1}{n}\right)$. Since the curve is increasing, the integral $<$ this sum $= U_n$ (AG).

Solution 5

(b)

Mark scheme

- Using left-endpoint rectangles, $L_n = \frac{1}{12}\left(1 - \frac{1}{n}\right)^2 + \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{12}\left(1 - \frac{1}{n}\right)\left(7 - \frac{1}{n}\right)$ (equivalently $L_n = U_n - \frac{4}{3n}$).

Solution 5

(c)

Mark scheme

- $U_n - L_n = \frac{1}{12}\left[\left(1+\frac{1}{n}\right)\left(7+\frac{1}{n}\right)-\left(1-\frac{1}{n}\right)\left(7-\frac{1}{n}\right)\right] = \frac{1}{12}\cdot\frac{16}{n} = \frac{4}{3n} \to 0$ as $n \to \infty$.

Question 1

9231/22 N25 ▪ Q1 ▪ 4 marks

The curve C has polar equation $r = e^{\frac{1}{4}\theta}$ for $0 \leqslant \theta \leqslant \pi$. Find the length of C. Give your answer in an exact form. **[4]**

Solution 1

Mark scheme

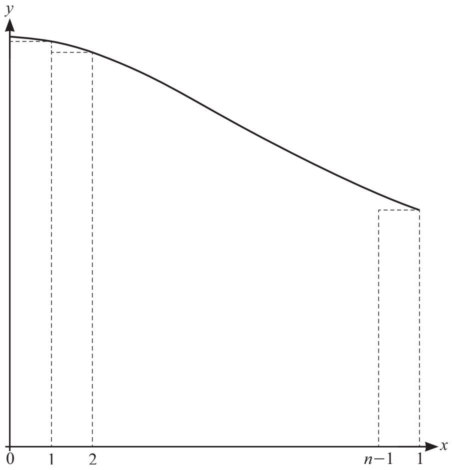
- $r = e^{\theta/4}$, $\dfrac{dr}{d\theta} = \frac{1}{4}e^{\theta/4}$, so $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{17}{16}e^{\theta/2}$.
  $$L = \int_0^{\pi} \frac{\sqrt{17}}{4} e^{\theta/4}\, d\theta = \sqrt{17}\left(e^{\pi/4} - 1\right).$$

Question 6

9231/22 N25 ▪ Q6 ▪ 13 marks

(a) Use the substitution $x = \frac{1}{2}\sqrt{2}\sinh u$ to find $\displaystyle\int \frac{1}{\sqrt{2x^2+1}}\,dx$. **[3]**

Question 6



Question 6

$$0 \qquad \frac{1}{n} \qquad \frac{2}{n} \qquad \qquad \frac{n-1}{n} \qquad 1 \qquad x$$

Question 6

9231/22 N25 ▪ Q6 ▪ 13 marks

(b) The diagram shows the curve with equation $y = \dfrac{1}{\sqrt{2x^2+1}}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\dfrac{1}{n}$. By considering the sum of the areas of these rectangles, show that $\displaystyle\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} < \tfrac{1}{2}\sqrt{2}\,\ln\left(\sqrt{2}+\sqrt{3}\right)$. **[5]**

Question 6

9231/22 N25 ▪ Q6 ▪ 13 marks

(c) Use a similar method to find, in terms of n, a lower bound for $\displaystyle\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2 + n^2}}$. **[4]**

Question 6

9231/22 N25 ▪ Q6 ▪ 13 marks

(d) Deduce the exact value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2 + n^2}}$. **[1]**

Solution 6

(a)

Mark scheme

- $x = \frac{1}{\sqrt{2}} \sinh u \Rightarrow 2x^2 + 1 = \cosh^2 u$, $dx = \frac{1}{\sqrt{2}} \cosh u \, du$, so the integral $= \displaystyle\int \frac{1}{\sqrt{2}} \, du = \frac{1}{\sqrt{2}} \sinh^{-1}(\sqrt{2}\, x) + C$.

Solution 6

(b) Mark scheme

- $\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} = \sum_{r=1}^{n} \frac{1}{n}\, y\left(\frac{r}{n}\right)$ is the right-endpoint Riemann sum; since y is decreasing it underestimates $\int_0^1 y\,dx = \frac{1}{\sqrt{2}} \sinh^{-1}\sqrt{2} = \frac{\sqrt{2}}{2}\ln(\sqrt{2}+\sqrt{3})$. Hence the sum $< \frac{1}{2}\sqrt{2}\,\ln(\sqrt{2}+\sqrt{3})$.

Solution 6

(c)

Mark scheme

- Since y decreases, $\frac{1}{n}y\left(\frac{r}{n}\right) \geqslant \int_{r/n}^{(r+1)/n} y\,dx$; summing gives the lower bound
$$\int_{1/n}^{(n+1)/n} y\,dx = \frac{\sqrt{2}}{2}\left[\sinh^{-1}\frac{\sqrt{2}(n+1)}{n} - \sinh^{-1}\frac{\sqrt{2}}{n}\right].$$

Solution 6

(d)

Mark scheme

- Both bounds tend to $\int_0^1 y\, dx$, so $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} = \frac{\sqrt{2}}{2} \ln(\sqrt{2}+\sqrt{3})$.

Question 1

9231/24 N25 ▪ Q1 ▪ 5 marks

It is given that $I_n = \int_0^1 x^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$, $I_n = \cosh 1 - n \sinh 1 + n(n-1) I_{n-2}$. **[3]**

(b) Find the exact value of I_2. **[2]**

Solution 1

(a)

Mark scheme

- Integrating by parts: $I_n = [x^n \cosh x]_0^1 - n \displaystyle\int_0^1 x^{n-1} \cosh x \, dx$. Again:
  $= [x^n \cosh x]_0^1 - n \left(\left[x^{n-1} \sinh x\right]_0^1 - (n-1) \int_0^1 x^{n-2} \sinh x \, dx \right) =$
  $\cosh 1 - n(\sinh 1 - (n-1) I_{n-2}) = \cosh 1 - n \sinh 1 + n(n-1) I_{n-2}$ (AG).

Solution 1

(b)

Mark scheme

- $I_2 = \cosh 1 - 2\sinh 1 + 2I_0$, where $I_0 = \int_0^1 \sinh x \, dx = \cosh 1 - 1$. So $I_2 = 3\cosh 1 - 2\sinh 1 - 2$ $\left(= \frac{1}{2}(\mathrm{e} + 5\mathrm{e}^{-1}) - 2\right)$.

Question 4

9231/24 N25 ▪ Q4 ▪ 10 marks

The diagram shows the curve with equation $y = \frac{1}{2}(3^x)$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.

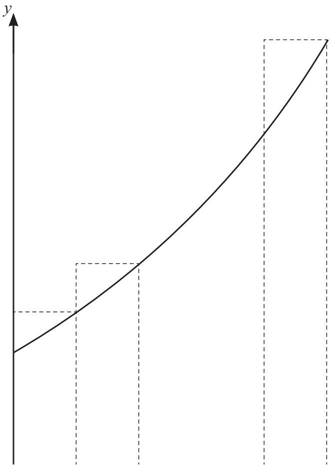
(a) By considering the sum of the areas of these rectangles, show that

$\int_0^1 \frac{1}{2}(3^x)\,dx < U_N$, where $U_N = \dfrac{3^{1/N}}{N(3^{1/N}-1)}$. **[4]**

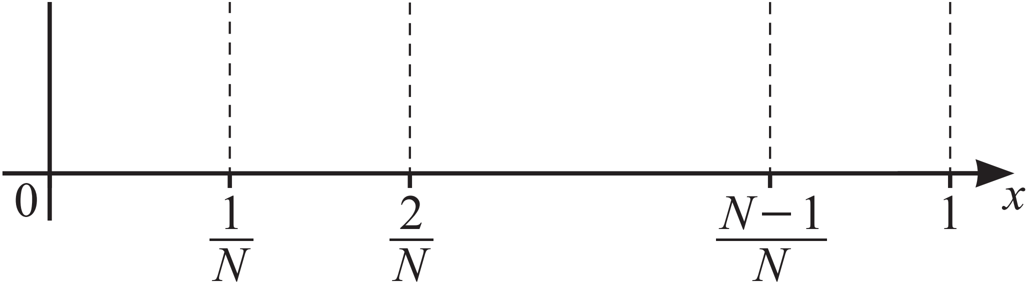
(b) Use a similar method to find, in terms of N, a lower bound L_N for $\int_0^1 \frac{1}{2}(3^x)\,dx$. **[4]**

(c) By simplifying $U_N - L_N$, show that $\lim_{N\to\infty}(U_N - L_N) = 0$. **[2]**

Question 4



Question 4



Solution 4

(a)

Mark scheme

- Right-endpoint sum:
$$\int_0^1 \tfrac{1}{2}(3^x)\,dx < \tfrac{1}{2}\sum_{n=1}^{N} \tfrac{1}{N}\left(3^{1/N}\right)^n = \tfrac{1}{2N}\cdot\frac{3^{1/N}(3-1)}{3^{1/N}-1} = \frac{3^{1/N}}{N(3^{1/N}-1)} = U_N$$
(AG).

Solution 4

(b)

Mark scheme

- Left-endpoint sum: $L_N = \frac{1}{2N}\sum_{n=0}^{N-1}\left(3^{1/N}\right)^n = \frac{1}{2N}\cdot\dfrac{3-1}{3^{1/N}-1} = \dfrac{1}{N(3^{1/N}-1)}$.

Solution 4

(c)

Mark scheme

- $U_N - L_N = \dfrac{3^{1/N}}{N(3^{1/N}-1)} - \dfrac{1}{N(3^{1/N}-1)} = \dfrac{3^{1/N}-1}{N(3^{1/N}-1)} = \dfrac{1}{N} \to 0$ as $N \to \infty$.

Question 2

9231/21 J25 ▪ Q2 ▪ 7 marks

Let $I_n = \int_0^1 (1-x)^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$, $I_n = -1 + n(n-1)I_{n-2}$. **[4]**

(b) Find the exact value of I_2. **[3]**

Solution 2

(a)

Mark scheme

- $I_n = [(1-x)^n \cosh x]_0^1 + n\int_0^1 (1-x)^{n-1}\cosh x\,dx =$
  $-1 + n\left([(1-x)^{n-1}\sinh x]_0^1 + (n-1)\int_0^1 (1-x)^{n-2}\sinh x\,dx\right)$. The bracket term is 0, so $I_n = -1 + n(n-1)I_{n-2}$ (AG).

Solution 2

(b)

Mark scheme

- $I_0 = \displaystyle\int_0^1 \sinh x \, dx = [\cosh x]_0^1 = \cosh 1 - 1$. So
  $I_2 = -1 + 2(1) I_0 = -1 + 2(\cosh 1 - 1) = 2\cosh 1 - 3 \ \left(= \mathrm{e} + \mathrm{e}^{-1} - 3\right)$.

Question 4

9231/21 J25 ▪ Q4 ▪ 9 marks

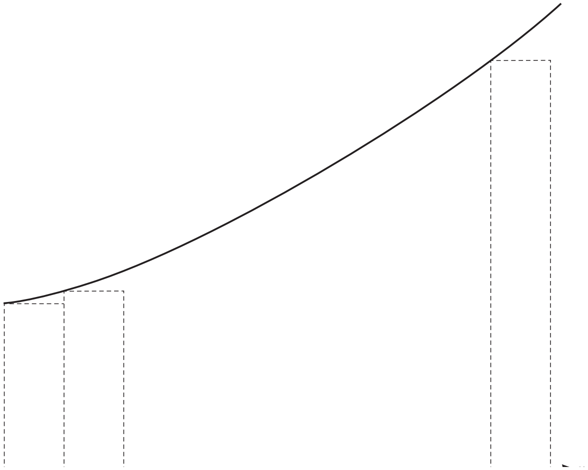
The diagram shows the curve with equation $y = \dfrac{1}{\sqrt{x}}\mathrm{e}^{\sqrt{x}}$ for $x \geqslant 1$, together with a set of $n-1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}\mathrm{e}^{\sqrt{r}} < \left(2 + \frac{1}{\sqrt{n}}\right)\mathrm{e}^{\sqrt{n}} - 2\mathrm{e}.$$ **[5]**

(b) Use a similar method to find, in terms of n, a lower bound for $\displaystyle\sum_{r=1}^{n} \frac{1}{\sqrt{r}}\mathrm{e}^{\sqrt{r}}$. **[4]**

Question 4



Question 4

Solution 4

(a) Mark scheme

- Left-endpoint rectangle sum (increasing curve):
  $\displaystyle\sum_{r=1}^{n-1} \frac{1}{\sqrt{r}} \mathrm{e}^{\sqrt{r}} < \int_1^n \frac{1}{\sqrt{x}} \mathrm{e}^{\sqrt{x}}\, dx = \left[2\mathrm{e}^{\sqrt{x}}\right]_1^n = 2\mathrm{e}^{\sqrt{n}} - 2\mathrm{e}.$ Adding the $r = n$ term:
  $\sum_{r=1}^{n} \dfrac{1}{\sqrt{r}} \mathrm{e}^{\sqrt{r}} < 2\mathrm{e}^{\sqrt{n}} - 2\mathrm{e} + \dfrac{1}{\sqrt{n}} \mathrm{e}^{\sqrt{n}} = \left(2 + \dfrac{1}{\sqrt{n}}\right) \mathrm{e}^{\sqrt{n}} - 2\mathrm{e}$ (AG).

Solution 4

(b) Mark scheme

- Right-endpoint rectangles give $\displaystyle\sum_{r=2}^{n} \frac{1}{\sqrt{r}} e^{\sqrt{r}} > \int_1^n \frac{1}{\sqrt{x}} e^{\sqrt{x}}\, dx = 2e^{\sqrt{n}} - 2e$. Adding the $r = 1$ term: $\sum_{r=1}^{n} \dfrac{1}{\sqrt{r}} e^{\sqrt{r}} > 2e^{\sqrt{n}} - 2e + e = 2e^{\sqrt{n}} - e$.

Question 6

9231/23 J25 ▪ Q6 ▪ 10 marks

The diagram shows the curve with equation $y = \dfrac{1}{x^2+1}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

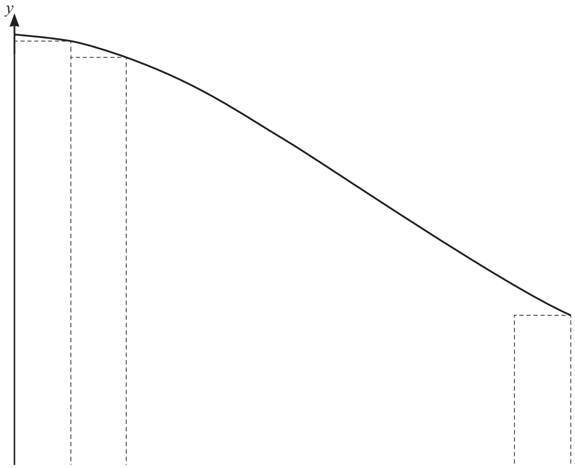
(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{n}{n^2+r^2} < \tfrac{1}{4}\pi.$$ **[5]**

(b) Use a similar method to find a lower bound for $\displaystyle\sum_{r=1}^{n} \frac{n}{n^2+r^2}$. Give your answer in terms of n and π. **[4]**

(c) Deduce the exact value of $\displaystyle\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$. **[1]**

Question 6



Question 6

Solution 6

(a)

Mark scheme

- Right-endpoint rectangle sum (decreasing curve):
$$\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(r/n)^2+1}=\sum_{r=1}^{n}\frac{n}{n^2+r^2}<\int_0^1\frac{1}{x^2+1}\,dx=\left[\tan^{-1}x\right]_0^1=\tfrac{1}{4}\pi \text{ (AG)}.$$

Solution 6

(b)

Mark scheme

- Left-endpoint rectangles (including $r = 0$, excluding $r = n$) overestimate:
  $\frac{1}{n} + \displaystyle\sum_{r=1}^{n} \frac{n}{n^2 + r^2} - \frac{1}{2n} \geqslant \frac{1}{4}\pi$, so $\sum_{r=1}^{n} \dfrac{n}{n^2 + r^2} \geqslant \frac{1}{4}\pi - \frac{1}{2n}$.

Solution 6

(c)

Mark scheme

- Both bounds $\to \frac{1}{4}\pi$ as $n \to \infty$, so $\displaystyle\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2 + r^2} = \frac{1}{4}\pi$.