Exercise walk-through

Integration ▪ 2.4

eXercise

You must be able to

- use $\int \frac{1}{a^2+x^2}\,dx$ and $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ (and completing the square)
- derive and use a reduction formula
- find an arc length

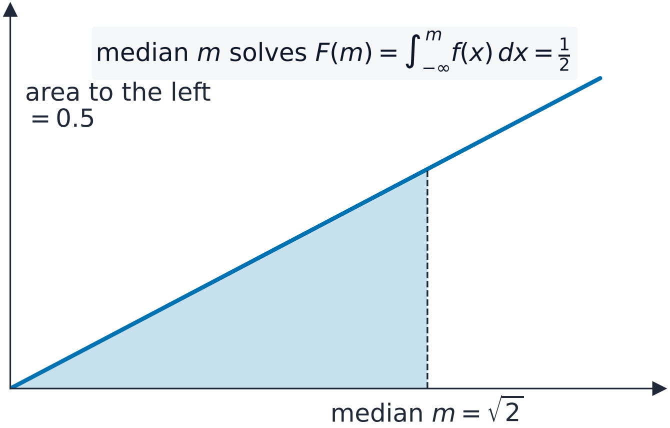
Method — Standard integrals & reduction

An integral is the area under a curve; the inverse-function forms handle $\frac{1}{a^2+x^2}$ *and* $\frac{1}{\sqrt{a^2-x^2}}$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C.$$

Reduction: $I_n = \int_0^{\pi/2} \sin^n x\,dx$ satisfies $I_n = \dfrac{n-1}{n} I_{n-2}$.

Method — Standard integrals & reduction



A region under a curve — the area is the integral

Practice 2.1 ▪ 2 marks

Find $\displaystyle\int \frac{1}{9+x^2}\,dx$.

Solution 2.1

Method: Standard integrals & reduction

Worked steps

$\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ **M1** **A1**.

Practice 2.2 ▪ 2 marks

Find $\displaystyle\int \frac{1}{\sqrt{25 - x^2}}\, dx$.

Solution 2.2

Method: Standard integrals & reduction

Worked steps

$\sin^{-1}\dfrac{x}{5} + C$ **M1** **A1**.

Exam-style 3.1 ▪ 1 mark

$\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$ equals:

A $\dfrac{\pi}{8}$

B $\dfrac{\pi}{4}$

C $\dfrac{\pi}{2}$

D π

Solution 3.1

Method: Standard integrals & reduction

A $\dfrac{\pi}{8}$ ✓

B $\dfrac{\pi}{4}$

C $\dfrac{\pi}{2}$

D π

Worked steps

A. $\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\tan^{-1}1 = \frac{1}{2}\cdot\frac{\pi}{4} = \dfrac{\pi}{8}.$

Exam-style 3.2 ▪ 4 marks

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$.

(a) Using integration by parts, show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$.

(b) Hence evaluate I_4.

Solution 3.2

Method: Standard integrals & reduction

Worked steps

(a) $I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$; by parts ($u = \sin^{n-1} x$, $dv = \sin x \, dx$):
$I_n = [-\sin^{n-1} x \cos x]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx$ **M1**; the boundary term is 0 **M1**; $\cos^2 x = 1 - \sin^2 x$ gives $I_n = (n-1)(I_{n-2} - I_n)$ **M1**; so $nI_n = (n-1)I_{n-2}$, i.e.
$I_n = \dfrac{n-1}{n} I_{n-2}$ **A1**.

(b) $I_0 = \dfrac{\pi}{2}$; $I_2 = \frac{1}{2} I_0 = \dfrac{\pi}{4}$; $I_4 = \frac{3}{4} I_2 = \dfrac{3\pi}{16}$ **M1** **M1** **A1**.

Exam-style 3.3 ▪ 4 marks

Find $\int \frac{1}{x^2 - 4x + 13}\,dx$ by completing the square.

Solution 3.3

Method: Standard integrals & reduction

Worked steps

$x^2 - 4x + 13 = (x-2)^2 + 9$ M1; $\displaystyle\int \frac{1}{(x-2)^2+9}\,dx = \tfrac{1}{3}\tan^{-1}\frac{x-2}{3} + C$ M1 M1 A1.