Past-paper walk-through

Differentiation ▪ 2.3

eXercise

Questions in this set

- 9231/21 N25 Q2 ▪ 8 marks
- 9231/22 N25 Q3 ▪ 9 marks
- 9231/22 N25 Q7 ▪ 11 marks
- 9231/24 N25 Q7 ▪ 13 marks
- 9231/23 J25 Q1 ▪ 5 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 2

9231/21 N25 ▪ Q2 ▪ 8 marks

The curve C has parametric equations $x = \sinh t$ and $y = t + \cosh t$.

(a) Find $\dfrac{dy}{dx}$ in terms of t. **[2]**

(b) Show that $\dfrac{d^2y}{dx^2} = \dfrac{1 - \sinh t}{\cosh^3 t}$. **[4]**

(c) Find the Maclaurin's series for y in terms of x up to and including the term in x^2. **[2]**

Solution 2

(a)

Mark scheme

- $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \sinh t}{\cosh t}.$$

Solution 2

(b) Mark scheme

- $\dfrac{d}{dt}\left(\dfrac{1+\sinh t}{\cosh t}\right) = \dfrac{\cosh t(\cosh t) - (1+\sinh t)\sinh t}{\cosh^2 t} = \dfrac{1-\sinh t}{\cosh^2 t}$. Then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\dfrac{dt}{dx} = \dfrac{1-\sinh t}{\cosh^2 t}\cdot\dfrac{1}{\cosh t} = \dfrac{1-\sinh t}{\cosh^3 t}$ (AG).

Solution 2

(c)

Mark scheme

- At $t = 0$: $x = \sinh 0 = 0$, $y = 0 + \cosh 0 = 1$, $y'(0) = \dfrac{1+0}{1} = 1$, $y''(0) = \dfrac{1-0}{1} = 1$. So $y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) = 1 + x + \frac{1}{2}x^2$.

Question 3

9231/22 N25 ▪ Q3 ▪ 9 marks

The variables x and y are such that $y = 2$ when $x = -1$ and $x^2y + (x + y)^3 = 3$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{1}{4}$ when $x = -1$. **[3]**

(b) Find the value of $\dfrac{d^2y}{dx^2}$ when $x = -1$. **[6]**

Solution 3

(a)

Mark scheme

- Differentiating: $2xy + x^2y' + 3(x+y)^2(1+y') = 0$. At $(-1, 2)$ (so $x + y = 1$): $-4 + y' + 3(1+y') = 0 \Rightarrow 4y' = 1 \Rightarrow y' = \frac{1}{4}$.

Solution 3

(b)

Mark scheme

- Differentiating again: $2y + 4xy' + x^2y'' + 6(x+y)(1+y')^2 + 3(x+y)^2y'' = 0$.
  At $(-1, 2)$, $y' = \frac{1}{4}$:
  $4 - 1 + y'' + 6\left(\frac{5}{4}\right)^2 + 3y'' = 0 \Rightarrow \frac{99}{8} + 4y'' = 0 \Rightarrow y'' = -\frac{99}{32}$.

Question 7

9231/22 N25 ▪ Q7 ▪ 11 marks

(a) Show that $\dfrac{d}{dx}\left(\frac{1}{2}x\sqrt{4-x^2}+2\sin^{-1}\left(\frac{1}{2}x\right)\right)=\sqrt{4-x^2}$. **[3]**

(b) Find the solution of the differential equation $2\dfrac{dy}{dx}+\dfrac{y}{2+x}=2\sqrt{2-x}$ for which $y=\frac{1}{2}$ when $x=1$. Give your answer in an exact form. **[8]**

Solution 7

(a) Mark scheme

- Differentiating gives
  $\frac{1}{2}\sqrt{4-x^2} - \frac{x^2}{2\sqrt{4-x^2}} + \frac{2}{\sqrt{4-x^2}} = \frac{(4-x^2)-x^2+4}{2\sqrt{4-x^2}} = \frac{4-x^2}{\sqrt{4-x^2}} = \sqrt{4-x^2}$. AG.

Solution 7

(b) Mark scheme

- Integrating factor $\sqrt{2+x}$: $\frac{d}{dx}\left(y\sqrt{2+x}\right) = \sqrt{(2+x)(2-x)} = \sqrt{4-x^2}$. So $y\sqrt{2+x} = \frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$ (part (a)). At $x = 1$, $y = \frac{1}{2}$: $C = -\frac{\pi}{3}$. Hence $y = \dfrac{\frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} - \frac{\pi}{3}}{\sqrt{2+x}}$.

Question 7

9231/24 N25 ▪ Q7 ▪ 13 marks

The curve C has parametric equations $x = 8\ln(\tan\frac{1}{2}t) - 3\cot t - 3t$, $y = 6\ln(\tan\frac{1}{2}t) + 4\cot t + 4t$, for $\frac{1}{6}\pi \leqslant t \leqslant \frac{1}{3}\pi$.

(a)(i) Show that $\dfrac{dx}{dt} = 8\operatorname{cosec} t + 3\cot^2 t$. **[1]**

(a)(ii) Find $\dfrac{dy}{dt}$. **[1]**

(a)(iii) Find the exact value of the length of C. **[6]**

(b) Show that $\dfrac{d^2y}{dx^2} = \dfrac{a\cot t\operatorname{cosec} t(\operatorname{cosec}^2 t + 1)}{(b\operatorname{cosec} t + c\cot^2 t)^n}$, where a, b, c and n are integers to be determined. **[5]**

Solution 7

(a)(i)

Mark scheme

- $\dfrac{dx}{dt} = 8 \cdot \dfrac{\frac{1}{2}\sec^2 \frac{1}{2}t}{\tan \frac{1}{2}t} + 3\,\mathrm{cosec}^2 t - 3 = \dfrac{8}{\sin t} + 3\,\mathrm{cosec}^2 t - 3 =$
  $8\,\mathrm{cosec}\, t + 3(\mathrm{cosec}^2 t - 1) = 8\,\mathrm{cosec}\, t + 3\cot^2 t$ (AG).

Solution 7

(a)(ii)

Mark scheme

- $\dfrac{dy}{dt} = 6\operatorname{cosec} t - 4\operatorname{cosec}^2 t + 4 = 6\operatorname{cosec} t - 4(\operatorname{cosec}^2 t - 1) = 6\operatorname{cosec} t - 4\cot^2 t.$

Solution 7

(a)(iii)

Mark scheme

- $\dot{x}^2 + \dot{y}^2 = (8\operatorname{cosec} t + 3\cot^2 t)^2 + (6\operatorname{cosec} t - 4\cot^2 t)^2 = 100\operatorname{cosec}^2 t + 25\cot^4 t = 25(\operatorname{cosec}^2 t + 1)^2$. Length $= \displaystyle\int_{\pi/6}^{\pi/3} 5(\operatorname{cosec}^2 t + 1)\, dt = 5\left[-\cot t + t\right]_{\pi/6}^{\pi/3} = 5\left(-\frac{1}{\sqrt{3}} + \frac{1}{3}\pi + \sqrt{3} - \frac{1}{6}\pi\right) = \frac{5}{6}\pi + \frac{10}{3}\sqrt{3}$.

Solution 7

(b) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{6\,\text{cosec}\,t - 4\cot^2 t}{8\,\text{cosec}\,t + 3\cot^2 t}$. Differentiating with respect to t and multiplying by $\dfrac{dt}{dx}$:
  $\dfrac{d^2y}{dx^2} = \dfrac{50\cot t\,\text{cosec}\,t(\text{cosec}^2 t + 1)}{(8\,\text{cosec}\,t + 3\cot^2 t)^3}$, so $a = 50,\ b = 8,\ c = 3,\ n = 3$.

Question 1

9231/23 J25 ▪ Q1 ▪ 5 marks

Find the Maclaurin's series for $\mathrm{e}^{\left(\frac{1}{x+2}\right)}$ up to and including the term in x^2. **[5]**

Solution 1

Mark scheme

- With $f(x) = \mathrm{e}^{(x+2)^{-1}}$: $f'(x) = -(x+2)^{-2}\mathrm{e}^{(x+2)^{-1}}$ and $f''(x) = (x+2)^{-4}\mathrm{e}^{(x+2)^{-1}} + 2(x+2)^{-3}\mathrm{e}^{(x+2)^{-1}} = \dfrac{2x+5}{(x+2)^4}\mathrm{e}^{(x+2)^{-1}}$. At $x = 0$: $f(0) = \mathrm{e}^{1/2}$, $f'(0) = -\frac{1}{4}\mathrm{e}^{1/2}$, $f''(0) = \frac{5}{16}\mathrm{e}^{1/2}$. So $\mathrm{e}^{(x+2)^{-1}} = \mathrm{e}^{1/2} - \frac{1}{4}\mathrm{e}^{1/2}x + \frac{5}{32}\mathrm{e}^{1/2}x^2$.