Past-paper walk-through

Matrices ▪ 2.2

eXercise

Questions in this set

- 9231/21 N25 Q1 ▪ 3 marks
- 9231/21 N25 Q6 ▪ 10 marks
- 9231/22 N25 Q8 ▪ 15 marks
- 9231/24 N25 Q8 ▪ 13 marks
- 9231/21 J25 Q8 ▪ 13 marks
- 9231/23 J25 Q8 ▪ 14 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 1

9231/21 N25 ▪ Q1 ▪ 3 marks

Find the values of k for which the system of equations $x - y + z = 0$, $x + ky + 3z = 0$, $x + 2y + kz = 0$ does not have a unique solution. **[3]**

Solution 1

Mark scheme

- $\det\begin{pmatrix}1 & -1 & 1\\ 1 & k & 3\\ 1 & 2 & k\end{pmatrix} = (k^2-6)+(k-3)+(2-k) = k^2-7$. For no unique solution the determinant is 0: $k^2-7=0 \Rightarrow k=\pm\sqrt{7}$.

Question 6

9231/21 N25 ▪ Q6 ▪ 10 marks

The matrix $\mathbf{P}$ has non-zero eigenvalues and is given by $\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}$.

(a) State, in terms of a, the eigenvalues of $\mathbf{P}$. **[1]**

(b)(i) Find $\mathbf{P}^2$ in terms of a. **[1]**

(b)(ii) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$ in terms of a. **[3]**

(c) The 3×3 matrix $\mathbf{A}$ has eigenvalues 1, 2, 3 with corresponding eigenvectors $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix}$ respectively. Find $\mathbf{A}$ in terms of a. **[5]**

Solution 6

(a)

Mark scheme

- The eigenvalues are the diagonal entries (upper triangular): a, $2a$, $-3a$.

Solution 6

(b)(i)

Mark scheme

- $\mathbf{P}^2 = \begin{pmatrix} a^2 & 3a & -2a-1 \\ 0 & 4a^2 & a \\ 0 & 0 & 9a^2 \end{pmatrix}$.

(b)(ii)

Mark scheme

- Characteristic equation: $\mathbf{P}^3 - 7a^2\mathbf{P} + 6a^3\mathbf{I} = \mathbf{0}$. Multiplying by $\mathbf{P}^{-1}$:
  $6a^3\mathbf{P}^{-1} = 7a^2\mathbf{I} - \mathbf{P}^2$, so $\mathbf{P}^{-1} = \dfrac{1}{6a^3}\begin{pmatrix} 6a^2 & -3a & 2a+1 \\ 0 & 3a^2 & -a \\ 0 & 0 & -2a^2 \end{pmatrix}$.

Solution 6

(c)

Mark scheme

- With $\mathbf{D} = \mathrm{diag}(1, 2, 3)$ and $\mathbf{P}$ the matrix of eigenvectors (the same $\mathbf{P}$),

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{pmatrix} 1 & \frac{1}{2a} & -\frac{4a+1}{6a^2} \\ 0 & 2 & \frac{1}{3a} \\ 0 & 0 & 3 \end{pmatrix}.$$

Question 8

9231/22 N25 ▪ Q8 ▪ 15 marks

(a) Find the values of k for which the system of equations $x - y + 2kz = 1$, $kx + y + 2z = 2$, $2x - y + z = 3$ does not have a unique solution. **[3]**

(b) Given that $k = -\frac{1}{2}$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. **[3]**

(c) Given instead that $k = -1$, show that the system of equations in part (a) is also inconsistent. Interpret this situation geometrically. **[4]**

(d) The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$. Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A}$, where p and q are integers to be determined. **[5]**

Solution 8

(a)

Mark scheme

- $\det\begin{pmatrix}1 & -1 & 2k\\ k & 1 & 2\\ 2 & -1 & 1\end{pmatrix} = -2k^2-3k-1=0 \Rightarrow 2k^2+3k+1=0 \Rightarrow k=-\frac{1}{2}$ or $k=-1$.

Solution 8

(b)

Mark scheme

- With $k = -\frac{1}{2}$, eliminating variables leads to $0 = 2$, a contradiction, so the system is inconsistent. Geometrically the three planes have no common point — they form a triangular prism (their pairwise lines of intersection are parallel).

Solution 8

(c)

Mark scheme

- With $k = -1$, equations 1 and 2 reduce to $x - y - 2z = 1$ and $x - y - 2z = -2$: these are parallel, distinct planes, so there is no common point and the system is inconsistent.

Solution 8

(d)

Mark scheme

- The characteristic equation is $\lambda^3 - 3\lambda^2 + 8\lambda = 0$, so by Cayley–Hamilton $\mathbf{A}^3 = 3\mathbf{A}^2 - 8\mathbf{A}$. Then $\mathbf{A}^4 = 3\mathbf{A}^3 - 8\mathbf{A}^2 = 3(3\mathbf{A}^2 - 8\mathbf{A}) - 8\mathbf{A}^2 = \mathbf{A}^2 - 24\mathbf{A}$, so $p = 1$, $q = -24$.

Question 8

9231/24 N25 ▪ Q8 ▪ 13 marks

(a) Find the set of values of a for which the system of equations $5x + ay = 3$, $20x - 5y = 2$, $2x - 3y + z = 1$ has a unique solution and interpret this situation geometrically. **[3]**

(b) Given that $a = -\frac{5}{4}$, show that the system of equations in part (a) is inconsistent and interpret this situation geometrically. **[3]**

(c) The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{pmatrix}$. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{PDP}^{-1}$. **[7]**

Solution 8

(a)

Mark scheme

- $\det\begin{pmatrix} 5 & a & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{pmatrix} = -25 - 20a$. For a unique solution this is non-zero: $a \neq -\frac{5}{4}$. Geometrically, the three planes intersect at a single point.

Solution 8

(b)

Mark scheme

- With $a = -\frac{5}{4}$: $5x - \frac{5}{4}y = 3$ gives $20x - 5y = 12$, contradicting $20x - 5y = 2$ ($2 = 12$). So the system is inconsistent: the first two planes are distinct parallel planes, and the third is not parallel to them (no common point).

Solution 8

(c)

Mark scheme

- Eigenvalues of $\mathbf{A}$ are $5, -5, 1$ with eigenvectors $(1, 2, -1)$, $(0, 2, 1)$, $(0, 0, 1)$. $(\mathbf{A} + 3\mathbf{I})^2$ has the same eigenvectors with eigenvalues $8^2, (-2)^2, 4^2 = 64, 4, 16$. So $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 64 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16 \end{pmatrix}$.

Question 8

9231/21 J25 ▪ Q8 ▪ 13 marks

(a) It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$. Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^3$ with corresponding eigenvalue λ^3. **[2]**

(b) The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \end{pmatrix}$. Show that the eigenvalues of $\mathbf{A}$ are -1, 1 and 5. **[2]**

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} - 2\mathbf{I} = \mathbf{PDP}^{-1}$. **[6]**

(d) Use the characteristic equation of $\mathbf{A}$ to show that $(\mathbf{A} - 2\mathbf{I})^3 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$, where a, b and c are constants to be determined. **[3]**

Solution 8

(a)

Mark scheme

- $\mathbf{Ae} = \lambda\mathbf{e}$. Then $\mathbf{A}^3\mathbf{e} = \mathbf{A}^2(\mathbf{Ae}) = \mathbf{A}^2(\lambda\mathbf{e}) = \lambda\mathbf{A}(\mathbf{Ae}) = \lambda\mathbf{A}(\lambda\mathbf{e}) = \lambda^2(\mathbf{Ae}) = \lambda^3\mathbf{e}$. So $\mathbf{e}$ is an eigenvector of $\mathbf{A}^3$ with eigenvalue λ^3.

Solution 8

(b)

Mark scheme

- $\det(\mathbf{A} - \lambda\mathbf{I}) = (-1-\lambda)\big((1-\lambda)(5-\lambda) - 0\big) - 3(0) + 4(0) = (-1-\lambda)(1-\lambda)(5-\lambda) = 0$, so $\lambda = -1, 1, 5$ (AG).

Solution 8

(c)

Mark scheme

- Eigenvectors of $\mathbf{A}$: $\lambda = -1 \to (1,0,0)$; $\lambda = 1 \to (5,2,1)$; $\lambda = 5 \to (2,0,3)$. $\mathbf{A} - 2\mathbf{I}$ has the same eigenvectors with eigenvalues $-3, -1, 3$. So $\mathbf{P} = \begin{pmatrix} 1 & 5 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Solution 8

(d)

Mark scheme

- $(\mathbf{A}-2\mathbf{I})^3 = \mathbf{A}^3 - 6\mathbf{A}^2 + 12\mathbf{A} - 8\mathbf{I}$. The characteristic equation gives $\mathbf{A}^3 = 5\mathbf{A}^2 + \mathbf{A} - 5\mathbf{I}$, so $(\mathbf{A}-2\mathbf{I})^3 = 5\mathbf{A}^2 + \mathbf{A} - 5\mathbf{I} - 6\mathbf{A}^2 + 12\mathbf{A} - 8\mathbf{I} = -\mathbf{A}^2 + 13\mathbf{A} - 13\mathbf{I}$ ($a = -1$, $b = 13$, $c = -13$).

Question 8

9231/23 J25 ▪ Q8 ▪ 14 marks

(a) Find the values of a for which the system of equations $\frac{3}{2}x + 3y + 8z = 1$, $ax + 3y + 4z = 2$, $ay - z = 3$ does not have a unique solution. **[3]**

(b) The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 3 & 8 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{pmatrix}$. Given that $\mathbf{B} = \mathbf{A}^{-1}$, use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{B}^2 = p\mathbf{I} + q\mathbf{A}$, where p and q are constants to be determined. **[4]**

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. **[7]**

Solution 8

(a)

Mark scheme

- $\det \begin{pmatrix} \frac{3}{2} & 3 & 8 \\ a & 3 & 4 \\ 0 & a & -1 \end{pmatrix} = 0 \Rightarrow 8a^2 - 3a - \frac{9}{2} = 0$, so $a = \frac{3}{16}\left(1 \pm \sqrt{17}\right)$.

Solution 8

(b)

Mark scheme

- Characteristic equation $\left(\lambda - \frac{3}{2}\right)(\lambda - 3)(\lambda + 1) = \lambda^3 - \frac{7}{2}\lambda^2 + \frac{9}{2} = 0$, so $\frac{9}{2}\mathbf{I} = \frac{7}{2}\mathbf{A}^2 - \mathbf{A}^3$. Multiplying by $\mathbf{B}^2 = \mathbf{A}^{-2}$: $\frac{9}{2}\mathbf{B}^2 = \frac{7}{2}\mathbf{I} - \mathbf{A}$, so $\mathbf{B}^2 = \frac{7}{9}\mathbf{I} - \frac{2}{9}\mathbf{A}$ ($p = \frac{7}{9}$, $q = -\frac{2}{9}$).

Solution 8

(c)

Mark scheme

- Eigenvalues of $\mathbf{A}$ are $\frac{3}{2}, 3, -1$ with eigenvectors $(1, 0, 0)$, $(2, 1, 0)$, $(2, 1, -1)$. $\mathbf{A}^{-1}$ has the same eigenvectors with reciprocal eigenvalues $\frac{2}{3}, \frac{1}{3}, -1$. So $\mathbf{P} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}$.