Past-paper walk-through

Hyperbolic functions ▪ 2.1

eXercise

Questions in this set

- 9231/21 N25 Q7 ▪ 12 marks
- 9231/24 N25 Q2 ▪ 6 marks
- 9231/21 J25 Q6 ▪ 15 marks
- 9231/23 J25 Q2 ▪ 8 marks
- 9231/23 J25 Q3 ▪ 9 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 7

9231/21 N25 ▪ Q7 ▪ 12 marks

(a) Show that $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$. **[3]**

(b) Find the solution of the differential equation $x\dfrac{dy}{dx} - y = x^2 \tanh^{-1} x$, for $0 < x < 1$, given that $y = 0$ when $x = \frac{1}{2}$. Give your answer in an exact form. **[9]**

Solution 7

(a) Mark scheme

- Using $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$:
  $$\frac{d}{dx}\left(\tanh^{-1} x\right) = \tfrac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} = \tfrac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2} = \frac{1}{1-x^2}$$
  (AG). (Alternatively $\tanh y = x \Rightarrow \operatorname{sech}^2 y\,\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$.)

Solution 7

(b) Mark scheme

- $\dfrac{dy}{dx} - \dfrac{y}{x} = x \tanh^{-1} x$; integrating factor $\mathrm{e}^{-\int x^{-1} dx} = x^{-1}$, so $\dfrac{d}{dx}(x^{-1}y) = \tanh^{-1} x$. Then $x^{-1}y = x\tanh^{-1} x - \displaystyle\int \frac{x}{1-x^2}\,dx = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$. Using $y = 0$ at $x = \frac{1}{2}$: $C = -\frac{1}{4}\ln\frac{27}{16}$. So $x^{-1}y = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) - \frac{1}{4}\ln\frac{27}{16}$ (i.e. $y = x\left(x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) - \frac{1}{4}\ln\frac{27}{16}\right)$).

Question 2

9231/24 N25 ▪ Q2 ▪ 6 marks

(a) Find the exact values of x for which $\cosh 2x = 6\sinh^2 x$, giving your answers in logarithmic form. **[4]**

(b) Sketch the curves $C_1 : y = \cosh 2x$ and $C_2 : y = 6\sinh^2 x$ on the same diagram. **[2]**

Solution 2

(a)

Mark scheme

- $\cosh 2x = 1 + 2\sinh^2 x = 6\sinh^2 x \Rightarrow 4\sinh^2 x = 1 \Rightarrow \sinh x = \pm\frac{1}{2}$. So $x = \sinh^{-1}(\pm\frac{1}{2}) = \ln\left(\pm\frac{1}{2} + \sqrt{\frac{1}{4} + 1}\right) = \ln(\pm\frac{1}{2} + \frac{1}{2}\sqrt{5})$.

Solution 2

(b)

Mark scheme

- $C_1 : y = \cosh 2x$ is a $\cosh$ curve with minimum $(0, 1)$; $C_2 : y = 6\sinh^2 x$ has minimum $(0, 0)$ and is steeper. They intersect where $\sinh x = \pm\frac{1}{2}$ (at $y = \frac{3}{2}$).

Question 6

9231/21 J25 ▪ Q6 ▪ 15 marks

(a) Starting from the definitions of $\tanh$ and sech in terms of exponentials, prove that $1 - \tanh^2 u = \operatorname{sech}^2 u$. **[3]**

(b) Show that $\dfrac{d}{dt}(\operatorname{sech}^{-1} t) = -\dfrac{1}{t\sqrt{1-t^2}}$. **[4]**

(c) It is given that $x = \tanh^{-1} t$ and $y = t\operatorname{sech}^{-1} t$, for $0 < t < 1$. Show that $\dfrac{dy}{dx} = -\sqrt{1-t^2} + (1-t^2)\operatorname{sech}^{-1} t$. **[4]**

(d) Find $\dfrac{d^2y}{dx^2}$ in terms of t. **[4]**

Solution 6

(a)

Mark scheme

- $\tanh u = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$ and $\operatorname{sech} u = \dfrac{2}{e^u + e^{-u}}$. Then
  $1 - \tanh^2 u = \dfrac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} = \dfrac{4}{(e^u + e^{-u})^2} = \operatorname{sech}^2 u$ (AG).

Solution 6

(b) Mark scheme

- $u = \operatorname{sech}^{-1} t \Rightarrow \operatorname{sech} u = t$. Differentiating implicitly: $-\tanh u \operatorname{sech} u \dfrac{du}{dt} = 1$, so $\dfrac{du}{dt} = -\dfrac{1}{\tanh u \operatorname{sech} u} = -\dfrac{1}{t\sqrt{1-t^2}}$ (using part (a), $\tanh u = \sqrt{1-t^2}$) (AG).

Solution 6

(c) Mark scheme

- $\dfrac{dy}{dt} = \operatorname{sech}^{-1} t + t\left(-\dfrac{1}{t\sqrt{1-t^2}}\right) = \operatorname{sech}^{-1} t - \dfrac{1}{\sqrt{1-t^2}}$ and $\dfrac{dx}{dt} = \dfrac{1}{1-t^2}$. So $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \left(\operatorname{sech}^{-1} t - \dfrac{1}{\sqrt{1-t^2}}\right)(1-t^2) = -\sqrt{1-t^2} + (1-t^2)\operatorname{sech}^{-1} t$ (AG).

Solution 6

(d) Mark scheme

- $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{t}{\sqrt{1-t^2}} - \dfrac{1-t^2}{t\sqrt{1-t^2}} - 2t\,\mathrm{sech}^{-1}t$. Multiplying by $\dfrac{dt}{dx} = 1-t^2$: $\dfrac{d^2y}{dx^2} = t\sqrt{1-t^2} - \dfrac{(1-t^2)^{3/2}}{t} - 2t(1-t^2)\mathrm{sech}^{-1}t$.

Question 2

9231/23 J25 ▪ Q2 ▪ 8 marks

(a) Starting from the definitions of $\tanh$ and sech in terms of exponentials, prove that $\tanh^2 t + \operatorname{sech}^2 t = 1$. **[3]**

(b) The curve C has parametric equations $x = \ln(\cosh t)$, $y = \tan^{-1}(\sinh t)$, for $0 \leqslant t \leqslant 1$. Find the length of C. **[5]**

Solution 2

(a) Mark scheme

- $\tanh t = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$, $\operatorname{sech} t = \dfrac{2}{e^t + e^{-t}}$. Then
  $\tanh^2 t + \operatorname{sech}^2 t = \dfrac{(e^t - e^{-t})^2 + 4}{(e^t + e^{-t})^2} = \dfrac{e^{2t} + e^{-2t} + 2}{(e^t + e^{-t})^2} = \dfrac{(e^t + e^{-t})^2}{(e^t + e^{-t})^2} = 1$ (AG).

Solution 2

(b)

Mark scheme

- $\dfrac{dx}{dt} = \tanh t$ and $\dfrac{dy}{dt} = \dfrac{\cosh t}{1+\sinh^2 t} = \operatorname{sech} t$ (using $1+\sinh^2 t = \cosh^2 t$).
  Length $= \displaystyle\int_0^1 \sqrt{\tanh^2 t + \operatorname{sech}^2 t}\, dt = \int_0^1 1\, dt = 1.$

Question 3

9231/23 J25 ▪ Q3 ▪ 9 marks

The curve C has equation $9y^2 - 3\sinh^{-1}(xy) = 1 - 3\ln 3$.

(a) Show that, at the point $\left(4, \frac{1}{3}\right)$ on C, $\dfrac{dy}{dx} = -\frac{1}{2}$. **[4]**

(b) Find the value of $\dfrac{d^2y}{dx^2}$ at the point $\left(4, \frac{1}{3}\right)$. **[5]**

Solution 3

(a) Mark scheme

- $\dfrac{d}{dx}(9y^2) = 18yy'$ and $\dfrac{d}{dx}\left(-3\sinh^{-1}(xy)\right) = -3 \cdot \dfrac{xy' + y}{\sqrt{1 + x^2y^2}}$. So $18yy' - \dfrac{3(xy' + y)}{\sqrt{1 + x^2y^2}} = 0$. At $\left(4, \frac{1}{3}\right)$, $\sqrt{1 + x^2y^2} = \frac{5}{3}$, giving $\frac{18}{3}y' - 3 \cdot \dfrac{4y' + \frac{1}{3}}{\frac{5}{3}} = 0 \Rightarrow y' = -\frac{1}{2}$ (AG).

Solution 3

(b) Mark scheme

- Differentiating again: $18yy'' + 18(y')^2 - 3\,\dfrac{d}{dx}\left(\dfrac{xy' + y}{\sqrt{1 + x^2y^2}}\right) = 0$. Substituting $(4, \frac{1}{3})$ and $y' = -\frac{1}{2}$ gives $-\frac{6}{5}y'' + \frac{87}{10} = 0 \Rightarrow y'' = \frac{29}{4}$.