# Exercise walk-through

## Proof by induction ▪ 1.7

eXercise

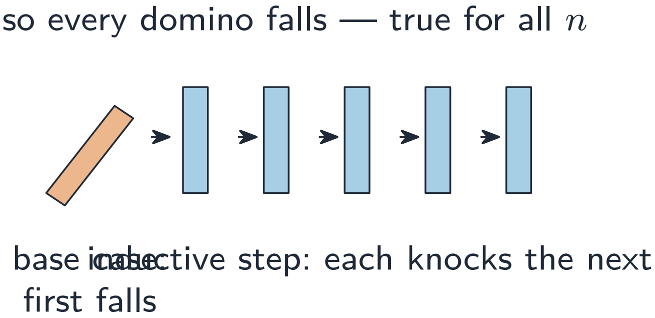
# You must be able to

- prove a summation formula by induction
- prove a divisibility result by induction
- write the base case, inductive step and conclusion clearly

# Method — The two steps

**Prove** $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$**.** Base $n = 1$: $1 = \frac{1}{2}(1)(2)$ ✓. Assume true for $k$; then $\sum_{1}^{k+1} = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$ — the formula at $n = k+1$. So true for all $n$.

# Method — The two steps




*A row of dominoes: the first falls, each knocks the next*

# Practice 2.1 ▪ 2 marks

State the two steps of a proof by induction.

# Solution 2.1

Method: The two steps

**Worked steps**

base case (true for $n = 1$) **B1**; inductive step (assume $n = k$, prove $n = k + 1$) **B1**.

# Practice 2.2 ▪ 1 mark

Verify the base case $n = 1$ for $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

# Solution 2.2

Method: The two steps

**Worked steps**

LHS $= 1$, RHS $= \frac{1}{6}(1)(2)(3) = 1$ ✓ **B1**.

# Exam-style 3.1 ▪ 1 mark

In an inductive proof, the inductive step assumes the result for $n = k$ and proves it for:

**A** $n = 1$

**B** $n = k - 1$

**C** $n = k + 1$

**D** all $n$ at once

# Solution 3.1

Method: The two steps

**A** $n = 1$

**B** $n = k - 1$

**C** $n = k + 1$ ✓

**D** all $n$ at once

## Worked steps

**C**. The inductive step proves the result for $n = k + 1$.

# Exam-style 3.2 ▪ 5 marks

Prove by induction that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

# Solution 3.2

Method: The two steps

**Worked steps**

base $n = 1$: LHS $= 2$, RHS $= \frac{1}{3}(1)(2)(3) = 2$ ✓ **B1**; assume $\sum_1^k r(r+1) = \frac{1}{3}k(k+1)(k+2)$ **M1**; $\sum_1^{k+1} = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$ **M1**; $= (k+1)(k+2)\left(\frac{k}{3}+1\right) = \frac{1}{3}(k+1)(k+2)(k+3)$ **M1**; this is the formula at $n = k+1$, so true for all $n$ **A1**.

# Exam-style 3.3 ▪ 5 marks

Prove by induction that $5^n - 1$ is divisible by $4$ for all positive integers $n$.

# Solution 3.3

Method: The two steps

**Worked steps**

base $n = 1$: $5^1 - 1 = 4$, divisible by $4$ ✓ **B1**; assume $5^k - 1 = 4m$ for integer $m$ **M1**; $5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(4m + 1) - 1 = 20m + 4 = 4(5m + 1)$ **M1** **M1**; divisible by $4$, so by induction true for all $n$ **A1**.