Past-paper walk-through

Polar coordinates ▪ 1.5

eXercise

Questions in this set

- 9231/11 N25 Q6 ▪ 15 marks
- 9231/12 N25 Q5 ▪ 12 marks
- 9231/14 N25 Q6 ▪ 12 marks
- 9231/11 J25 Q5 ▪ 13 marks
- 9231/13 J25 Q7 ▪ 16 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

9231/11 N25 ▪ Q6 ▪ 15 marks

The curve C has polar equation $r = \sin 3\theta$, for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(a) Sketch C and state the equation of the line of symmetry. **[3]**

(b) Find the exact value of the area of the region enclosed by C. **[4]**

(c) (You may use the identity $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$.) Find the maximum distance of a point on C from the initial line. **[5]**

(d) Find a Cartesian equation for C. **[3]**

Solution 6

(a)

Mark scheme

- A single loop from the pole in the first quadrant; line of symmetry $\theta = \frac{1}{6}\pi$.

Solution 6

(b)

Mark scheme

- Area $= \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{1}{4}\int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{1}{12}\pi.$

Solution 6

(c)

Mark scheme

- $y = r\sin\theta = (3\sin\theta - 4\sin^3\theta)\sin\theta = 3\sin^2\theta - 4\sin^4\theta$; $\frac{dy}{d\theta} = 0 \Rightarrow \sin^2\theta = \frac{3}{8}$, giving maximum distance $y = \frac{9}{16}$.

Solution 6

(d)

Mark scheme

- Using $y = r\sin\theta$ and $r^2 = x^2 + y^2$: $(x^2 + y^2)^2 = y(3x^2 - y^2)$.

Question 5

9231/12 N25 ▪ Q5 ▪ 12 marks

The curve C has polar equation $r^2 = \tan 2\theta$, where $0 \leqslant \theta \leqslant \frac{1}{8}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. **[2]**

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{8}\pi$. **[4]**

(c) Show that C has Cartesian equation $x^4 - 2xy - y^4 = 0$, given that $0 \leqslant x \leqslant \cos\left(\frac{1}{8}\pi\right)$ and $0 \leqslant y \leqslant \sin\left(\frac{1}{8}\pi\right)$. **[4]**

(d) Using your answer to (b), deduce the exact value of the area bounded by C, the x-axis and the line $x = \cos\left(\frac{1}{8}\pi\right)$. **[2]**

Solution 5

(a)

Mark scheme

- At $\theta = \frac{1}{8}\pi$, $r^2 = \tan\frac{1}{4}\pi = 1$, so the greatest distance from the pole is 1. The curve runs from the pole at $\theta = 0$ out to $r = 1$ at $\theta = \frac{1}{8}\pi$.

Solution 5

(b)

Mark scheme

- Area $= \frac{1}{2}\int_0^{\pi/8} \tan 2\theta \, d\theta = \frac{1}{2}\left[-\frac{1}{2}\ln\cos 2\theta\right]_0^{\pi/8} = \frac{1}{8}\ln 2.$

Solution 5

(c)

Mark scheme

- With $x = r\cos\theta$, $y = r\sin\theta$: $r^2\cos 2\theta = \sin 2\theta \Rightarrow r^2(x^2 - y^2) = 2xy \Rightarrow (x^2 + y^2)(x^2 - y^2) = 2xy \Rightarrow x^4 - 2xy - y^4 = 0$. AG.

Solution 5

(d)

Mark scheme

- The curve and the half-line bound a region inside triangle OPQ where $P = (\cos\frac{1}{8}\pi, \sin\frac{1}{8}\pi)$, $Q = (\cos\frac{1}{8}\pi, 0)$. Required area $= \frac{1}{2}\cos\frac{1}{8}\pi\sin\frac{1}{8}\pi - \frac{1}{8}\ln 2 = \frac{\sqrt{2}}{8} - \frac{1}{8}\ln 2 = \frac{1}{8}(\sqrt{2} - \ln 2)$.

Question 6

9231/14 N25 ▪ Q6 ▪ 12 marks

The curve C has polar equation $r = \cos\frac{1}{2}\theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C. **[2]**

(b) Find the exact value of the area of the region enclosed by C and the initial line. **[4]**

(c) Find the maximum distance of a point on C from the initial line, giving your answer in exact form. **[6]**

Solution 6

(a)

Mark scheme

- A single arc joining the pole O to the point $(1, 0)$, lying in the first two quadrants (above the initial line), with correct shape at the extremities and r decreasing from 1 to 0 as θ increases from 0 to π.

Solution 6

(b)

Mark scheme

- Area $= \frac{1}{2}\int_0^{\pi} \cos^2 \frac{1}{2}\theta \, d\theta = \frac{1}{4}\int_0^{\pi} (\cos\theta + 1)\, d\theta = \frac{1}{4}[\sin\theta + \theta]_0^{\pi} = \frac{1}{4}\pi.$

Solution 6

(c)

Mark scheme

- Distance from initial line $y = r\sin\theta = \cos\frac{1}{2}\theta\sin\theta$. $\dfrac{dy}{d\theta} = 0$: $\cos\frac{1}{2}\theta\cos\theta - \frac{1}{2}\sin\theta\sin\frac{1}{2}\theta = 0 \Rightarrow 3\cos^2\frac{1}{2}\theta - 2 = 0 \Rightarrow \cos^2\frac{1}{2}\theta = \frac{2}{3}$. Then $y = 2\cos^2\frac{1}{2}\theta\sin\frac{1}{2}\theta = 2\left(\frac{2}{3}\right)\sqrt{1-\frac{2}{3}} = \frac{4}{3}\sqrt{\frac{1}{3}} = \frac{4}{9}\sqrt{3}$.

Question 5

9231/11 J25 ▪ Q5 ▪ 13 marks

The curve C has polar equation $r = \theta\,\mathrm{e}^{\frac{1}{8}\theta}$, for $0 \leqslant \theta \leqslant 2\pi$.

(a) Sketch C. **[2]**

(b) Find the area of the region bounded by C and the initial line, giving your answer in the form $(p\pi^2 + q\pi + r)\mathrm{e}^{\frac{1}{2}\pi} + s$, where p, q, r and s are integers to be determined. **[6]**

(c) Show that, at the point of C furthest from the initial line, $\theta\cos\theta + \left(\frac{1}{8}\theta + 1\right)\sin\theta = 0$, and verify that this equation has a root between 5 and 5.05. **[5]**

Solution 5

(a)

Mark scheme

- A spiral on $[0, 2\pi]$ starting at the pole, r strictly increasing, with the correct orientation and one further intersection with the initial line.

Solution 5

(b)

Mark scheme

- Area $= \frac{1}{2}\int_0^{2\pi} \theta^2 e^{\frac{1}{4}\theta}\, d\theta$. Integrating by parts twice:

  $= \left[2\theta^2 e^{\frac{1}{4}\theta} - 16\theta e^{\frac{1}{4}\theta} + 64 e^{\frac{1}{4}\theta}\right]_0^{2\pi} = (8\pi^2 - 32\pi + 64)e^{\frac{1}{2}\pi} - 64.$

Solution 5

(c)

Mark scheme

- $y = \theta e^{\frac{1}{8}\theta} \sin\theta$. $\dfrac{dy}{d\theta} = \theta e^{\frac{1}{8}\theta}\cos\theta + \sin\theta\left(\frac{1}{8}\theta e^{\frac{1}{8}\theta} + e^{\frac{1}{8}\theta}\right)$; since $e^{\frac{1}{8}\theta} \neq 0$, setting $\frac{dy}{d\theta} = 0$ gives $\theta\cos\theta + \left(\frac{1}{8}\theta + 1\right)\sin\theta = 0$ (AG). At $\theta = 5$: $5\cos 5 + \left(\frac{5}{8} + 1\right)\sin 5 = -0.1399$; at $\theta = 5.05$: $+0.1336$. The sign change confirms a root between 5 and 5.05.

Question 7

9231/13 J25 ▪ Q7 ▪ 16 marks

The curve C has polar equation $r^2 = \mathrm{e}^{\sin\theta}\cos\theta$, for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole, giving your answers correct to 3 decimal places. **[5]**

(b) Find the polar coordinates of the point on C that is furthest from the half-line $\theta = \frac{1}{2}\pi$, giving your answers correct to 3 decimal places. **[5]**

(c) Sketch C. **[3]**

(d) Find the area of the region bounded by C, giving your answer in exact form. **[3]**

Solution 7

(a)

Mark scheme

- $r^2 = e^{\sin\theta}\cos\theta$; $\dfrac{d(r^2)}{d\theta} = e^{\sin\theta}(\cos^2\theta - \sin\theta) = 0 \Rightarrow \cos^2\theta - \sin\theta = 0 \Rightarrow \sin^2\theta + \sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2}(-1+\sqrt{5})$. Furthest point from the pole: $(r,\theta) = (1.208,\ 0.666)$.

Solution 7

(b)

Mark scheme

- Distance from the half-line $\theta = \frac{1}{2}\pi$ is $x = r\cos\theta = e^{\frac{1}{2}\sin\theta}\cos^{3/2}\theta$. $\dfrac{dx}{d\theta} = 0 \Rightarrow -3\sin\theta + \cos^2\theta = 0 \Rightarrow \sin^2\theta + 3\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2}(-3+\sqrt{13})$.
  Point: $(r, \theta) = (1.136,\ 0.308)$.

Solution 7

(c)

Mark scheme

- A closed loop passing through O and the point $(1, 0)$, lying in the first and fourth quadrants (more in the first), approaching the line $\theta = \frac{1}{2}\pi$.

Solution 7

(d)

Mark scheme

- Area $= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{\sin\theta} \cos\theta \, d\theta = \frac{1}{2} \left[\mathrm{e}^{\sin\theta} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2} \left(\mathrm{e} - \mathrm{e}^{-1} \right).$