Past-paper walk-through

Matrices ▪ 1.4

eXercise

Questions in this set

- 9231/11 N25 Q2 ▪ 11 marks
- 9231/12 N25 Q4 ▪ 10 marks
- 9231/11 J25 Q4 ▪ 9 marks
- 9231/13 J25 Q1 ▪ 11 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 2

9231/11 N25 ▪ Q2 ▪ 11 marks

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$.

(a) Give full details of the geometrical transformation in the *x*-*y* plane represented by $\mathbf{A}$. **[2]**

(b) Give full details of the geometrical transformation in the *x*-*y* plane represented by $\mathbf{B}$. **[2]**

(c) The triangle *DEF* in the *x*-*y* plane is transformed by $\mathbf{AB}$ onto triangle *PQR*. Show that the triangles *DEF* and *PQR* have the same area. **[2]**

(d) Find the equations of the invariant lines, through the origin, of the transformation in the *x*-*y* plane represented by $\mathbf{AB}$. **[5]**

Solution 2

(a)

Mark scheme

- Shear with the x-axis invariant (fixed), e.g. $(0, 1) \mapsto \left(\frac{3}{2}, 1\right)$.

Solution 2

(b)

Mark scheme

- Shear with the y-axis invariant (fixed), e.g. $(1,0) \mapsto \left(1, \frac{3}{2}\right)$.

Solution 2

(c)

Mark scheme

- $\mathbf{AB} = \begin{pmatrix} \frac{13}{4} & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$ and $\det \mathbf{AB} = \det \mathbf{A} \, \det \mathbf{B} = 1 \times 1 = 1$, so the area is unchanged.

Solution 2

(d)

Mark scheme

- $\frac{3}{2}x + my = m\left(\frac{13}{4}x + \frac{3}{2}my\right) \Rightarrow m^2 + \frac{3}{2}m - 1 = 0 \Rightarrow m = \frac{1}{2}$ or -2. Invariant lines: $y = -2x$ and $y = \frac{1}{2}x$.

Question 4

9231/12 N25 ▪ Q4 ▪ 10 marks

Let k and m be non-zero constants. The matrices are $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} k & 0 \\ 0 & m \end{pmatrix}$

and $\mathbf{C} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(a)(i) $m = 1$ **[2]**

(a)(ii) $m = k$ **[2]**

(b) Show that the matrix $\mathbf{ABC}$ is singular. **[6]**

Solution 4

(a)(i)

Mark scheme

- $\mathbf{B} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$: a stretch parallel to the x-axis with scale factor k (the y-coordinate is unchanged).

(a)(ii)

Mark scheme

- $\mathbf{B} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = k\mathbf{I}$: an enlargement centred on the origin with scale factor k.

Solution 4

(b)

Mark scheme

- $\mathbf{AB} = \begin{pmatrix} 0 & m \\ -k & m \\ k & m \end{pmatrix}$; the 3×3 product $\mathbf{ABC}$ passes through the 2×2 matrix $\mathbf{B}$, so $\operatorname{rank}(\mathbf{ABC}) \leqslant 2 < 3$, giving $\det(\mathbf{ABC}) = 0$ — i.e. $\mathbf{ABC}$ is singular.

Question 4

9231/11 J25 ▪ Q4 ▪ 9 marks

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $0 < \theta < 2\pi$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations in the *x*-*y* plane. State the type of each transformation, and make clear the order in which they are applied. **[2]**

(b) Find the value of θ for which the transformation represented by $\mathbf{M}$ has a line of invariant points. **[7]**

Solution 4

(a)

Mark scheme

- A rotation followed by a shear: an (anticlockwise) rotation through θ about the origin, then a shear with the *x*-axis fixed (with $(0,1) \mapsto (2,1)$).

Solution 4

(b)

Mark scheme

- $\mathbf{M} = \begin{pmatrix} \cos\theta + 2\sin\theta & 2\cos\theta - \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Invariant points satisfy $x\sin\theta + y\cos\theta = y$ and $x\cos\theta + 2x\sin\theta - y\sin\theta + 2y\cos\theta = x$. Eliminating leads to $\sin\theta + \cos\theta = 1$, so $\theta = \frac{1}{2}\pi$.

Question 1

9231/13 J25 ▪ Q1 ▪ 11 marks

The matrix $\mathbf{M}$ represents the sequence of two transformations in the x-y plane given by a stretch parallel to the x-axis, scale factor 14, followed by a rotation anticlockwise about the origin through angle $\frac{1}{3}\pi$.

(a) Show that $2\mathbf{M} = \begin{pmatrix} 14 & -\sqrt{3} \\ 14\sqrt{3} & 1 \end{pmatrix}$. **[4]**

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by $\mathbf{M}$. **[5]**

(c) The unit square S in the x-y plane is transformed by $\mathbf{M}$ onto the rectangle P. Find the matrix which transforms P onto S. **[2]**

Solution 1

(a) Mark scheme

- Stretch $= \begin{pmatrix} 14 & 0 \\ 0 & 1 \end{pmatrix}$, rotation $= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$. In the correct order,
  $\mathbf{M} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 14 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -\frac{\sqrt{3}}{2} \\ 7\sqrt{3} & \frac{1}{2} \end{pmatrix}$, so $2\mathbf{M} = \begin{pmatrix} 14 & -\sqrt{3} \\ 14\sqrt{3} & 1 \end{pmatrix}$ (AG).

Solution 1

(b)

Mark scheme

- For $y = mx$: $7\sqrt{3}x + \frac{1}{2}mx = m\left(7x - \frac{\sqrt{3}}{2}mx\right)$, giving $\sqrt{3}m^2 - 13m + 14\sqrt{3} = 0$, so $m = 2\sqrt{3}$ or $m = \frac{7}{\sqrt{3}}$. Invariant lines: $y = 2\sqrt{3}\,x$ and $y = \frac{7}{\sqrt{3}}x$.

Solution 1

(c)

Mark scheme

- The required matrix is $\mathbf{M}^{-1} = \frac{1}{14}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -7\sqrt{3} & 7 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{1}{14} & \frac{\sqrt{3}}{14} \\ -\sqrt{3} & 1 \end{pmatrix}$ (the inverse of $\mathbf{M}$).