# Past-paper walk-through

## Rational functions and graphs ▪ 1.2

eXercise

# Questions in this set

- 9231/11 N25 Q7 ▪ 14 marks
- 9231/12 N25 Q7 ▪ 14 marks
- 9231/14 N25 Q7 ▪ 16 marks
- 9231/11 J25 Q7 ▪ 15 marks
- 9231/13 J25 Q6 ▪ 14 marks

# Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

# Question 7

9231/11 N25 ▪ Q7 ▪ 14 marks

The curve $C$ has equation $y = \dfrac{x+2}{x^2+3x+1}$.

**(a)** Find the equations of the asymptotes of $C$. **[2]**

**(b)** Show that $C$ has no stationary points. **[3]**

**(c)** Sketch $C$, stating the coordinates of the intersections with the axes. **[3]**

**(d)** Sketch the curve with equation $y = \left|\dfrac{x+2}{x^2+3x+1}\right|$. **[2]**

**(e)** Find in exact form the set of values of $x$ for which $\left|\dfrac{x+2}{x^2+3x+1}\right| > 2$. **[4]**

# Solution 7

## (a)

**Mark scheme**

- Vertical asymptotes $x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$ (roots of $x^2 + 3x + 1 = 0$); horizontal asymptote $y = 0$.

# Solution 7

## (b)

**Mark scheme**

- $\frac{dy}{dx} = \dfrac{(x^2+3x+1)-(x+2)(2x+3)}{(x^2+3x+1)^2}$; setting the numerator to 0 gives $x^2+4x+5=0$, discriminant $16-20=-4<0$, no real roots — so no stationary points (AG).

# Solution 7

## (c)

**Mark scheme**

- Axis intersections $(0, 2)$ and $(-2, 0)$; vertical asymptotes at $x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$, curve $\to 0$ as $x \to \pm\infty$.

# Solution 7

**(d)**

**Mark scheme**

- Reflect the parts of $y = \frac{x+2}{x^2+3x+1}$ that lie below the $x$-axis in the $x$-axis; the curve touches 0 at $(-2, 0)$.

# Solution 7

## (e)

**Mark scheme**

- Critical points: $\dfrac{x+2}{x^2+3x+1} = 2 \Rightarrow x = -\frac{5}{2}, 0; = -2 \Rightarrow x = -\frac{7}{4} \pm \frac{1}{4}\sqrt{17}$. Solution: $-\frac{7}{4} - \frac{1}{4}\sqrt{17} < x < -\frac{5}{2}$ and $-\frac{7}{4} + \frac{1}{4}\sqrt{17} < x < 0$, excluding the asymptotes $x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.

# Question 7

9231/12 N25 ▪ Q7 ▪ 14 marks

The curve $C$ has equation $y = \dfrac{x^2 + x + 1}{x + 1}$.

**(a)** Find the equations of the asymptotes of $C$. **[3]**

**(b)** Find the coordinates of any stationary points on $C$. **[3]**

**(c)** Sketch $C$. **[3]**

**(d)** Sketch the curve with equation $y = \dfrac{|x|^2 + |x| + 1}{|x| + 1}$. **[2]**

**(e)** Find, in exact form, the set of values of $x$ for which $\dfrac{|x|^2 + |x| + 1}{|x| + 1} < 3$. **[3]**

# Solution 7

## (a)

**Mark scheme**

- $y = x + \dfrac{1}{x+1}$, so the asymptotes are $x = -1$ and $y = x$.

# Solution 7

**(b)**

**Mark scheme**

- $\dfrac{dy}{dx} = 1 - \dfrac{1}{(x+1)^2} = 0 \Rightarrow x = 0$ or $x = -2$. Stationary points $(0, 1)$ and $(-2, -3)$.

# Solution 7

**(c)**

**Mark scheme**

- Two branches with asymptotes $x = -1$ and $y = x$, a minimum at $(0, 1)$ and a maximum at $(-2, -3)$.

# Solution 7

**(d)**

**Mark scheme**

- An even curve (symmetric about the $y$-axis): the $x \geqslant 0$ branch of $C$ reflected in the $y$-axis, with minimum $(0, 1)$ and approaching $y = |x|$ for large $|x|$.

# Solution 7

## (e)

### Mark scheme

- Let $u = |x|$: $\dfrac{u^2 + u + 1}{u + 1} < 3 \Rightarrow u^2 - 2u - 2 < 0 \Rightarrow 0 \leqslant u < 1 + \sqrt{3}$. Hence $-(1 + \sqrt{3}) < x < 1 + \sqrt{3}$.

# Question 7

9231/14 N25 ▪ Q7 ▪ 16 marks

The curve $C$ has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

**(a)** Write down the equations of the asymptotes of $C$. **[3]**

# Question 7

9231/14 N25 ▪ Q7 ▪ 16 marks

The curve $C$ has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

**(b)** Show that $C$ has no stationary points. **[4]**

# Question 7

9231/14 N25 ▪ Q7 ▪ 16 marks

The curve $C$ has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

**(c)** Sketch $C$, stating the coordinates of any points of intersection with the axes. **[3]**

# Question 7

9231/14 N25 ▪ Q7 ▪ 16 marks

The curve $C$ has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

**(d)** On a separate diagram, sketch the curve with equation $y = \left|\dfrac{10x^2 - 11x - 18}{10x - 18}\right|$. **[2]**

# Question 7

9231/14 N25 ▪ Q7 ▪ 16 marks

The curve $C$ has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

**(e)** It is given that $\left|\dfrac{10x^2 - 11x - 18}{10x - 18}\right| < 4$ for $\dfrac{-29 - \sqrt{4441}}{20} < x < p$ and $\dfrac{-29 + \sqrt{4441}}{20} < x < q$. Find the exact values of $p$ and $q$. **[4]**

# Solution 7

## (a) Mark scheme

- $x = 1.8$ (where $10x - 18 = 0$). Dividing: $y = \dfrac{(10x-18)\left(x+\frac{7}{10}\right) - \frac{27}{5}}{10x-18} = x + \frac{7}{10} - \dfrac{27/5}{10x-18}$, so the oblique asymptote is $y = x + 0.7$.

# Solution 7

**(b)**

**Mark scheme**

- $\dfrac{dy}{dx} = \dfrac{100x^2 - 360x + 378}{(10x - 18)^2}$. Setting the numerator $= 0$: $50x^2 - 180x + 189 = 0$, discriminant $180^2 - 4(50)(189) = -5400 < 0$, so there are no real roots and hence no stationary points.

# Solution 7

## (c)

**Mark scheme**

- Intersections with the axes: $(0,1)$, $(-0.9,0)$ and $(2,0)$. Two branches with vertical asymptote $x=1.8$ and oblique asymptote $y=x+0.7$.

# Solution 7

**(d)**

**Mark scheme**

- Reflect in the $x$-axis the parts of $C$ lying below it (between the $x$-intercepts), giving sharp changes of direction at the $x$-axis intersections $(-0.9, 0)$ and $(2, 0)$.

# Solution 7

## (e)

### Mark scheme

- Solving $\dfrac{10x^2 - 11x - 18}{10x - 18} = 4$: $10x^2 - 51x + 54 = 0 \Rightarrow x = \frac{3}{2}$ or $x = \frac{18}{5}$.
  Hence $p = \frac{3}{2}$ and $q = \frac{18}{5}$.

# Question 7

9231/11 J25 ▪ Q7 ▪ 15 marks

The curve $C$ has equation $y = \dfrac{2x^2 - 5x}{2x^2 - 7x - 4}$.

**(a)** Find the equations of the asymptotes of $C$. **[2]**

**(b)** Find the coordinates of any stationary points on $C$. **[4]**

**(c)** Sketch $C$, stating the coordinates of the intersections with the axes. **[3]**

**(d)** Sketch the curve with equation $y = \left|\dfrac{2x^2 - 5x}{2x^2 - 7x - 4}\right|$. **[1]**

**(e)** Find in exact form the set of values of $x$ for which $\left|\dfrac{2x^2 - 5x}{2x^2 - 7x - 4}\right| < \dfrac{1}{9}$. **[5]**

# Solution 7

## (a)

**Mark scheme**

- Vertical asymptotes $x = -\frac{1}{2}$ and $x = 4$ (roots of $2x^2 - 7x - 4 = 0$); horizontal asymptote $y = 1$.

# Solution 7

## (b)

**Mark scheme**

- $\dfrac{dy}{dx} = \dfrac{(2x^2-7x-4)(4x-5)-(2x^2-5x)(4x-7)}{(2x^2-7x-4)^2}$. Numerator $=0$:
  $4x^2+16x-20=0 \Rightarrow x=1$ or $x=-5$. Stationary points $\left(-5, \frac{25}{27}\right)$ and $\left(1, \frac{1}{3}\right)$.

# Solution 7

## (c)

### Mark scheme

- Intersections with the axes: $(0,0)$ and $\left(\frac{5}{2},0\right)$. Curve has vertical asymptotes $x=-\frac{1}{2}$, $x=4$ and horizontal asymptote $y=1$.

# Solution 7

## (d)

**Mark scheme**

- Reflect in the $x$-axis the parts of $C$ lying below it; the curve touches the $x$-axis at $(0,0)$ and $\left(\frac{5}{2},0\right)$, with the same asymptotes (now $|y|$).

# Solution 7

## (e) Mark scheme

- Critical values from $\dfrac{2x^2-5x}{2x^2-7x-4} = \frac{1}{9}$ $(16x^2-38x+4=0 \Rightarrow x = \frac{19}{16} \pm \frac{3}{16}\sqrt{33})$ and $= -\frac{1}{9}$ $(20x^2-52x-4=0 \Rightarrow x = \frac{13}{10} \pm \frac{3}{10}\sqrt{21})$. Solution: $\frac{13}{10} - \frac{3}{10}\sqrt{21} < x < \frac{19}{16} - \frac{3}{16}\sqrt{33}$ and $\frac{19}{16} + \frac{3}{16}\sqrt{33} < x < \frac{13}{10} + \frac{3}{10}\sqrt{21}$.

# Question 6

9231/13 J25 ▪ Q6 ▪ 14 marks

The curve $C$ has equation $y = \dfrac{x^2 + a}{x + a}$, where $a$ is a positive constant.

**(a)** Find the equations of the asymptotes of $C$. **[3]**

**(b)** Find, in terms of $a$, the $x$-coordinates of the stationary points on $C$. **[3]**

**(c)** Sketch $C$, stating the coordinates of any intersections with the axes. **[3]**

**(d)** Sketch the curve with equation $y = \left|\dfrac{x^2 + a}{x + a}\right|$. **[1]**

**(e)** Find the set of values of $a$ for which $\left|\dfrac{x^2 + a}{x + a}\right| = a$ has two real solutions. **[4]**

# Solution 6

## (a)

**Mark scheme**

- Vertical asymptote $x = -a$. Dividing, $y = \dfrac{(x+a)(x-a) + a^2 + a}{x+a} = x - a + \dfrac{a^2 + a}{x+a}$, so the oblique asymptote is $y = x - a$.

# Solution 6

**(b)**

**Mark scheme**

- $\dfrac{dy}{dx} = 1 - (a^2 + a)(x + a)^{-2} = 0 \Rightarrow (x + a)^2 = a^2 + a \Rightarrow x = -a \pm \sqrt{a^2 + a}.$

# Solution 6

## (c)

**Mark scheme**

- Asymptotes $x = -a$ and $y = x - a$; $y$-intercept at $(0, 1)$. Two branches with a local maximum/minimum at $x = -a \pm \sqrt{a^2 + a}$.

# Solution 6

**(d)**

**Mark scheme**

- Reflect in the $x$-axis the parts of $C$ lying below it (following from the sketch in (c)), keeping the vertical asymptote $x = -a$.

# Solution 6

## (e)

**Mark scheme**

- Critical points: $\dfrac{x^2 + a}{x + a} = a \Rightarrow x^2 - ax + a - a^2 = 0$; the case $= -a$ gives $x^2 + ax + a + a^2 = 0$ which has no real roots $(a > 0)$. For two real solutions need the discriminant of $x^2 - ax + a - a^2 = 0$ positive: $a^2 - 4(a - a^2) = 5a^2 - 4a > 0 \Rightarrow a > \frac{4}{5}$.