Past-paper walk-through

Roots of polynomial equations ▪ 1.1

eXercise

Questions in this set

- 9231/11 N25 Q4 ▪ 8 marks
- 9231/12 N25 Q2 ▪ 8 marks
- 9231/14 N25 Q2 ▪ 9 marks
- 9231/11 J25 Q2 ▪ 7 marks
- 9231/13 J25 Q3 ▪ 9 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 4

9231/11 N25 ▪ Q4 ▪ 8 marks

The quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$ has roots α, β, γ, δ.

(a) Show that a quartic equation with roots $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$, $2\delta + 1$ is $y^4 - 2y^3 + 4y^2 + 2y + 11 = 0$. **[4]**

(b) The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n + (2\delta + 1)^n$ is denoted by S_n. Find the value of S_2. **[2]**

(c) Given that $S_3 = -22$, find the value of S_4. **[2]**

Solution 4

(a)

Mark scheme

- $y = 2x + 1 \Rightarrow x = \frac{1}{2}(y - 1)$; substituting into $x^4 + x^3 + x^2 + x + 1 = 0$ and simplifying gives $y^4 - 2y^3 + 4y^2 + 2y + 11 = 0$ (AG).

Solution 4

(b)

Mark scheme

- For the new quartic $S_1 = 2$ and $\sum$ pairs $= 4$, so $S_2 = S_1^2 - 2\sum \text{pairs} = 2^2 - 2(4) = -4$.

Solution 4

(c)

Mark scheme

- Newton's identity:
  $S_4 - 2S_3 + 4S_2 + 2S_1 + 44 = 0 \Rightarrow S_4 = 2(-22) - 4(-4) - 2(2) - 44 = -76.$

Question 2

9231/12 N25 ▪ Q2 ▪ 8 marks

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β and γ. It is given that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 3$ and $\alpha^4 + \beta^4 + \gamma^4 = 5$.

(a) Find the values of b and c. **[3]**

(b) Find the value of d. **[5]**

Solution 2

(a)

Mark scheme

- $\alpha + \beta + \gamma = -b = 2 \Rightarrow b = -2$.
  $\sum \alpha\beta = \frac{1}{2}\left[(\sum \alpha)^2 - \sum \alpha^2\right] = \frac{1}{2}(4-3) = \frac{1}{2} = c$.

Solution 2

(b)

Mark scheme

- Using Newton's identities with $e_1 = 2$, $e_2 = \frac{1}{2}$: solving via p_3 and $p_4 = 5$ gives $e_3 = \alpha\beta\gamma = -\frac{7}{16}$. Since $\alpha\beta\gamma = -d$, $d = \frac{7}{16}$.

Question 2

9231/14 N25 ▪ Q2 ▪ 9 marks

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β and γ. It is given that $\gamma = \dfrac{1}{\beta}$.

(a) Show that $\beta + \dfrac{1}{\beta} = d - b$. **[3]**

(b) Show that $\beta + \dfrac{1}{\beta} = \dfrac{1-c}{d}$. **[2]**

(c)(i) Find the value of d. **[2]**

(c)(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **[2]**

Solution 2

(a)

Mark scheme

- Since $\beta\gamma = \beta \cdot \frac{1}{\beta} = 1$ and $\alpha\beta\gamma = -d$, we have $\alpha = -d$. Using $\alpha + \beta + \gamma = -b$ with $\gamma = \frac{1}{\beta}$: $-d + \beta + \frac{1}{\beta} = -b$, so $\beta + \frac{1}{\beta} = d - b$ (AG).

Solution 2

(b)

Mark scheme

- Using $\alpha\beta + \beta\gamma + \gamma\alpha = c$ with $\gamma = \frac{1}{\beta}$ and $\alpha = -d$: $-d\beta + 1 - \frac{d}{\beta} = c$, so $-d\left(\beta + \frac{1}{\beta}\right) = c - 1$, giving $\beta + \frac{1}{\beta} = \dfrac{1-c}{d}$ (AG).

Solution 2

(c)(i)

Mark scheme

- With $b = 3$, $c = -3$, equating the results of (a) and (b): $d - 3 = \dfrac{1-(-3)}{d} = \dfrac{4}{d} \Rightarrow d^2 - 3d - 4 = 0 \Rightarrow (d-4)(d+1) = 0$. Since $d > 0$, $d = 4$.

(c)(ii)

Mark scheme

- $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = b^2 - 2c = 3^2 - 2(-3) = 15.$

Question 2

9231/11 J25 ▪ Q2 ▪ 7 marks

The cubic equation $x^3 + 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3 - 1$, $\beta^3 - 1$, $\gamma^3 - 1$. **[3]**

(b) Find the value of $(\alpha^3 - 1)^2 + (\beta^3 - 1)^2 + (\gamma^3 - 1)^2$. **[2]**

(c) Find the value of $(\alpha^3 - 1)^3 + (\beta^3 - 1)^3 + (\gamma^3 - 1)^3$. **[2]**

Solution 2

(a)

Mark scheme

- Let $y = x^3 - 1$, so $x = (y+1)^{1/3}$. From $x^3 + 2x + 1 = 0$:
  $2(y+1)^{1/3} = -y - 2 \Rightarrow 8(y+1) = -(y+2)^3$, giving $y^3 + 6y^2 + 20y + 16 = 0$.

Solution 2

(b)

Mark scheme

- For the new cubic, $\sum(\alpha^3 - 1) = -6$ and $\sum(\alpha^3 - 1)(\beta^3 - 1) = 20$. So $\sum(\alpha^3 - 1)^2 = (-6)^2 - 2(20) = -4$.

Solution 2

(c)

Mark scheme

- Using the new cubic ($y^3 + 6y^2 + 20y + 16 = 0$) and Newton's identities: $\sum(\alpha^3 - 1)^3 = -6(-4) - 20(-6) - 16(3) = 96$.

Question 3

9231/13 J25 ▪ Q3 ▪ 9 marks

The quartic equation $x^4 + 7x^2 + 3x + 22 = 0$ has roots α, β, γ, δ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. **[2]**

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. **[2]**

(c) Use standard results from the list of formulae (MF19) to find the value of

$$\sum_{r=1}^{10} \left((\alpha^2 + r)^2 + (\beta^2 + r)^2 + (\gamma^2 + r)^2 + (\delta^2 + r)^2\right).$$ **[5]**

Solution 3

(a)

Mark scheme

- $(\alpha+\beta+\gamma+\delta)^2 = \sum\alpha^2 + 2\sum\alpha\beta$, with $\sum\alpha = 0$ and $\sum\alpha\beta = 7$: $0 = \sum\alpha^2 + 2(7)$, so $\alpha^2+\beta^2+\gamma^2+\delta^2 = -14$.

Solution 3

(b)

Mark scheme

- Newton's identity for the quartic ($S_1 = 0$, $S_2 = -14$): $S_4 + 7S_2 + 3S_1 + 88 = 0 \Rightarrow S_4 = -7(-14) - 88 = 10$. So $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 10$.

Solution 3

(c)

Mark scheme

- $(r+\alpha^2)^2 = r^2 + 2\alpha^2 r + \alpha^4$, so the sum
  $= \sum_{r=1}^{10} (4r^2 + 2(\sum\alpha^2)r + \sum\alpha^4) = \sum_{r=1}^{10}(4r^2 - 28r + 10) =$
  $4 \cdot \frac{10}{6}(11)(21) - 28 \cdot \frac{10\cdot 11}{2} + 10(10) = 1540 - 1540 + 100 = 100.$