

Further Probability & Statistics

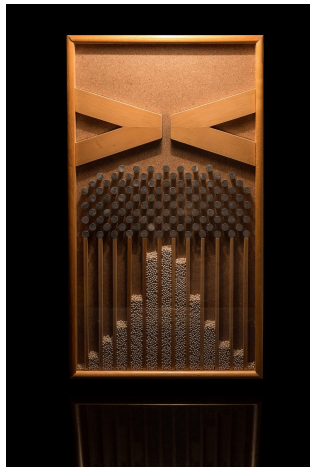
A-Level Further Mathematics ■ Topic 4

A-Level Further Mathematics



What we will cover

- 1 Continuous random variables
- 2 t -distribution inference
- 3 Chi-squared tests
- 4 Non-parametric tests
- 5 Generating functions
- 6 Recap





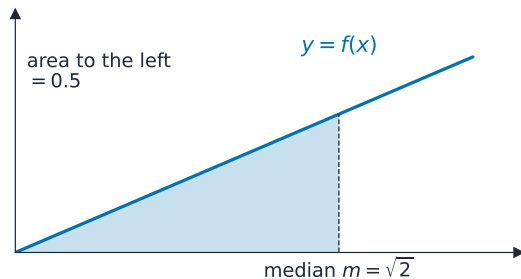
Continuous random variables

$$F(x) = P(X \leq x) = \int_{-\infty}^x f, \text{ and } f = F'.$$

$$f(x) = \frac{1}{2}x, 0 \leq x \leq 2$$

$$F = \frac{1}{4}x^2; \text{ median } m = \sqrt{2} = 1.41$$

Continuous random variables, in detail



median m solves $F(m) = \int_{-\infty}^m f(x) dx = \frac{1}{2}$

median: area 0.5 to its left

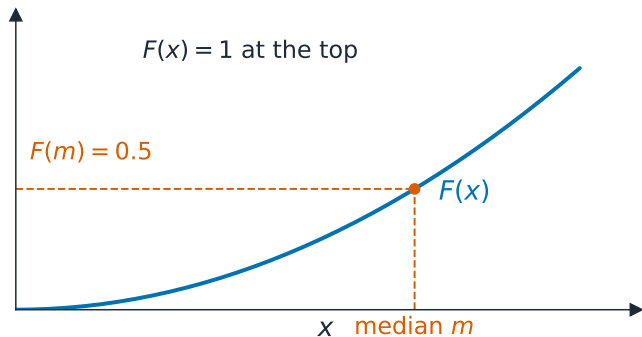
$$F(x) = P(X \leq x) = \int_{-\infty}^x f, \text{ and } f = F'.$$

The cumulative distribution function

cdf $F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$ – the running total of the density**going back** $f(x) = \frac{dF}{dx}$: differentiate the cdf to recover the density**its properties**non-decreasing, $F(-\infty) = 0$, $F(\infty) = 1$ **the median**solve $F(m) = 0.5$; **percentiles** solve $F(x) = p$ **why it matters**any probability is a difference of two values: $P(a < X \leq b) = F(b) - F(a)$

For a probability, reach for F ; for a mode or an expectation, reach for f . Knowing which one the question needs saves the whole calculation.

The cumulative graph $F(x)$ rises from 0



$$F(x) = P(X \leq x) = \int_{-\infty}^x f \cdot 0 \leq F \leq 1 \cdot \text{median: } F(m) = \frac{1}{2}$$

The cumulative graph $F(x)$ rises from 0 to 1; the height 0.5 is reached at the median.

The distribution of a related variable

the setup

X has a known distribution and $Y = g(X)$; find Y 's

the method

work through the **cumulative** function, never the density directly

for $Y = X^2$

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

then

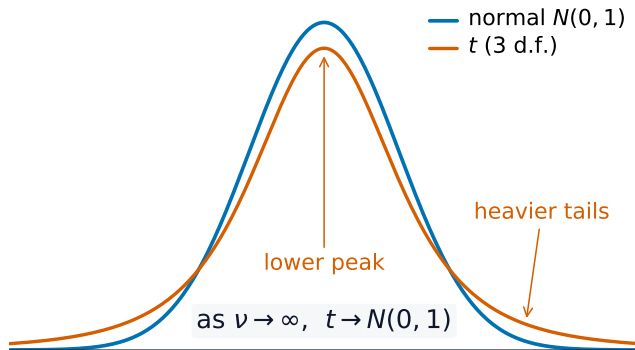
differentiate to get $f_Y(y)$

the caution

mind the **range** of Y , and whether g is one-to-one over X 's range

Never substitute into the density and hope. The cdf route is the only one that handles a many-to-one g correctly.

t -distribution inference



Small sample, unknown variance
 $\Rightarrow$ use the **t -distribution** t :

$$\bar{x} \pm t \frac{s}{\sqrt{n}} \quad (n - 1 \text{ d.f.}).$$

$$n = 10, \bar{x} = 50, s = 4, t = 2.262$$

$$50 \pm 2.86 \Rightarrow (47.1, 52.9)$$

When to use t rather than the normal

the condition

the sample is **small** and the population variance is **unknown**

the estimate

use the sample's own **unbiased estimate** $s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$

degrees of freedom

$\nu = n - 1$ for a single sample; the t -curve is wider than the normal and approaches it as ν grows

two samples

a **pooled estimate** of the common variance, with $\nu = n_1 + n_2 - 2$

confidence interval

$\bar{x} \pm t_{\nu} \frac{s}{\sqrt{n}}$ – same shape as the normal one, with t in place of z

The extra width of the t -curve is the price of estimating the variance from the same small sample. That is what makes the interval honest.



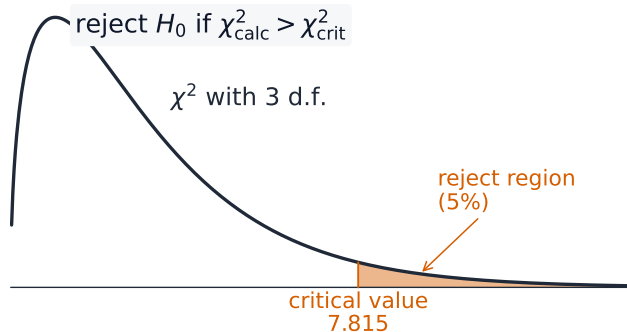
Chi-squared tests

Compare observed counts O with expected E :

$$\chi^2 = \sum \frac{(O - E)^2}{E}.$$

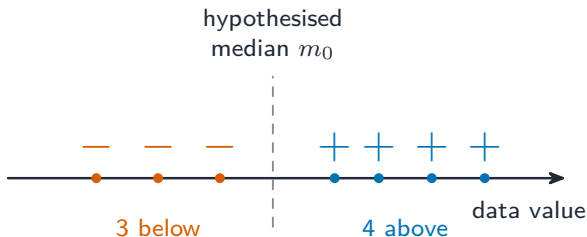
Large $\chi^2 \Rightarrow$ reject H_0 . Degrees of freedom = categories -1 , minus any estimated parameters.

Chi-squared tests, in detail



The test rejects the model when χ^2 exceeds the critical value, landing in the shaded 5%

Non-parametric tests



No assumption that the data is normal:

- **sign test** – counts above/below a median
- **Wilcoxon** signed-rank & rank-sum – use the sizes too

Worked example – a sign test

Ten values are tested against H_0 : median = 5. Nine fall above it. Is that evidence against H_0 ?

- under H_0 , each value is above with probability 0.5, so the count $X \sim B(10, 0.5)$
- $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + \binom{10}{10}(0.5)^{10}$
- $= \frac{10 + 1}{1024} = 0.0107$
- $0.0107 < 0.05$, so **reject** H_0 at the 5% level

The whole test is a binomial tail. Note it uses *only* the signs – discard how far above the median each value was, and a great deal of information with it.

The three non-parametric tests, and what each uses

sign test

uses only **which side** of the hypothesised median each value falls – the weakest, and the most assumption-free

Wilcoxon signed-rank

the **matched-pairs** test: uses the **sizes** of the differences as well as their signs, so it is more powerful

Wilcoxon rank-sum

for **two independent** samples – ranks all the data together and compares the rank totals

when to use them

when normality cannot be assumed, or the data are ordinal

goodness of fit

χ^2 compares observed with expected frequencies; a **contingency table** tests independence

Matched pairs → signed-rank. Two independent samples → rank-sum. Choosing the wrong one loses the whole question, so read for **paired** or **not** first.

Which test, and on what assumption

Hypothesis test

Every **hypothesis test** states H_0 , a test statistic, a significance level and a conclusion in context.

Parametric tests

t and χ^2 assume a **theoretical distribution** – normality, or expected counts of at least 5.

Wilcoxon signed-rank

The **Wilcoxon signed-rank test** : paired data, no normality assumed. Rank the differences and compare the rank sums.

Wilcoxon rank-sum

The **Wilcoxon rank-sum test** : two *independent* samples, again distribution-free.

A **probability density function** is what the parametric tests assume and the non-parametric ones avoid – which is why the latter are used on small or skewed samples.

Probability generating functions

$$G(t) = E(t^X) = \sum_x P(X=x) t^x.$$

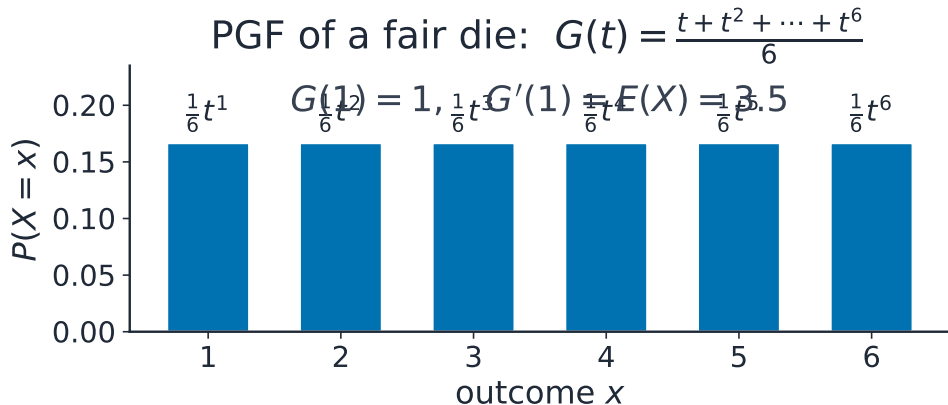
$E(X) = G'(1)$; the PGF of a sum of independent variables is the **product** of their PGFs.

$$P = 0.5, 0.3, 0.2 \text{ for } 0, 1, 2$$

$$G = 0.5 + 0.3t + 0.2t^2$$

$$E(X) = G'(1) = 0.7$$

PGF of a fair die packs equal



$$G(t) = E(t^X) \cdot E(X) = G'(1) \cdot \text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$$

PGF of a fair die packs equal masses at 1 through 6 into $G(t)$

Probability generating functions, continued



Dice: a starting point for probability and discrete random variables.

Exam tips

- For a **continuous random variable**, the pdf integrates to 1 over its range, and $E(X) = \int xf(x) dx$.
- Use the **t-distribution** when the sample is small and the population variance is unknown; state the degrees of freedom.
- For a **chi-squared** test compute $\sum (O - E)^2 / E$, compare with the critical value at the right degrees of freedom, and **combine classes with** $E < 5$.
- State H_0 and H_1 and give the conclusion **in context** for every test.



Topic 4 – key takeaways

1

Continuous

median solves $F(m) = 0.5$

2

t -inference

small samples use the t -distribution

3

χ^2

$\sum (O - E)^2 / E$; fit & independence

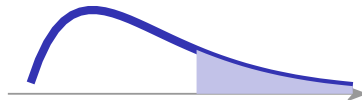
4

PGF

$G(t) = E(t^X)$; $E(X) = G'(1)$

Check yourself

- 1 Which distribution for a small-sample mean test?
- 2 Write the chi-squared statistic.
- 3 What does the sign test compare?
- 4 How do you get $E(X)$ from a PGF?



Answers 1. the t -distribution 2. $\sum (O - E)^2 / E$ 3. counts above/below a median
4. $E(X) = G'(1)$