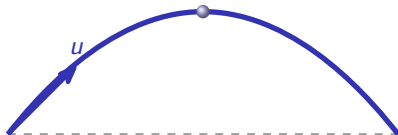


Further Mechanics

A-Level Further Mathematics ■ Topic 3

A-Level Further Mathematics



What we will cover

- 1 Projectiles
- 2 Rigid-body equilibrium
- 3 Circular motion
- 4 Hooke's law & elastic energy
- 5 Variable force & restitution
- 6 Recap





Projectile motion

Separate horizontal & vertical:

$$\text{range} = \frac{u^2 \sin 2\alpha}{g}, \quad H = \frac{u^2 \sin^2 \alpha}{2g}.$$

$$u = 20, \alpha = 30^\circ$$

$$\text{range} = 34.6 \text{ m}, \quad H = 5 \text{ m}$$

A projectile is two independent one-dimensional motions

the model

a **projectile** moves freely under gravity: **constant acceleration** g downwards, and **no** horizontal acceleration

horizontal

$x = u \cos \alpha \cdot t$ – constant velocity throughout

vertical

$y = u \sin \alpha \cdot t - \frac{1}{2}gt^2$ – constant acceleration

Cartesian equation

eliminate t : $y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$ – a parabola

bounding parabola

varying α at fixed u sweeps out an envelope; nothing beyond it is reachable at that speed

Treat the two directions **separately** and let *time* be the only quantity they share. Every projectile question is solved by finding t from one direction and using it in the other.

The trajectory, and the boundary of what it can reach

The trajectory

Eliminating t between the two component equations gives the Cartesian equation of the **trajectory** – the path – and it is a **parabola**.

Same speed, many angles

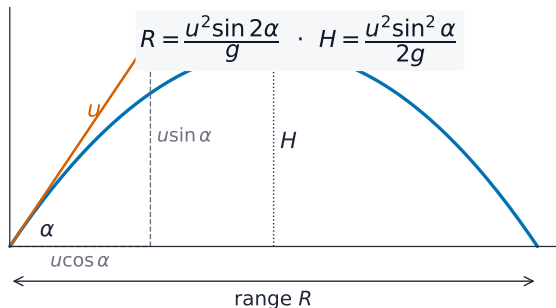
For one fixed launch speed, every trajectory lies under a **bounding parabola**: the envelope of everything reachable.

Which angle

Two launch angles reach the same range; 45° is the one that reaches furthest on level ground.

Write x and y separately, then eliminate t . The parabola is a *result*, never the starting point.

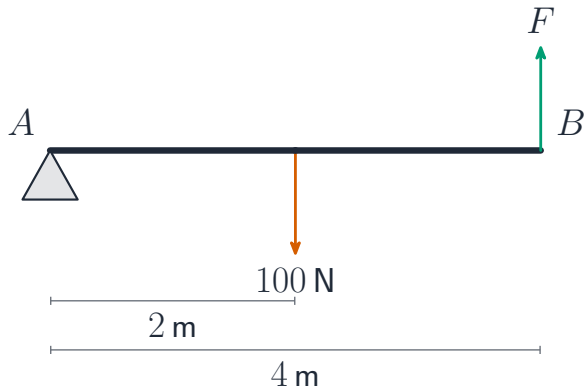
Projectile motion, in detail



$$\text{range} = \frac{u^2 \sin 2\alpha}{g}, \quad H = \frac{u^2 \sin^2 \alpha}{2g}$$

Treat horizontal and vertical motion **separately**: constant velocity across, constant acceleration down. Range is greatest at $\alpha = 45^\circ$.

Equilibrium of a rigid body



Moment = force $\times$
perpendicular distance; weight
acts at the **centre of mass**.

Equilibrium: $\sum F = 0$ and
 $\sum M = 0$.

Beam 4 m, 100 N, pivot at A

$$F \times 4 = 100 \times 2 \Rightarrow F = 50 \text{ N}$$

Centre of mass, and the two ways a body gives way

Centre of mass

The weight acts at the **centre of mass** . For a uniform flat shape – a **lamina** – find it by **symmetry** ; for a composite body, treat it as a set of particles.

Toppling

Toppling : the body turns over about an edge, once the line of the weight falls **outside** the base.

Sliding

Sliding : it slips instead, once the applied force exceeds the limiting friction.

Which comes first

Compare the two thresholds. The smaller one happens; an exam question usually asks which, and why.

Equilibrium needs **both** conditions: the vector sum of the **coplanar forces** is zero *and* the sum of the moments about any point is zero.

Equilibrium of a rigid body needs two conditions

condition 1

the **vector sum of the forces** is zero – so no translational acceleration

condition 2

the **sum of the moments** about any point is zero – so no rotational acceleration

coplanar forces

all in one plane, so condition 1 is two scalar equations and condition 2 is one

choosing the point

take moments about a point where an **unknown force acts** – it contributes no moment and drops out

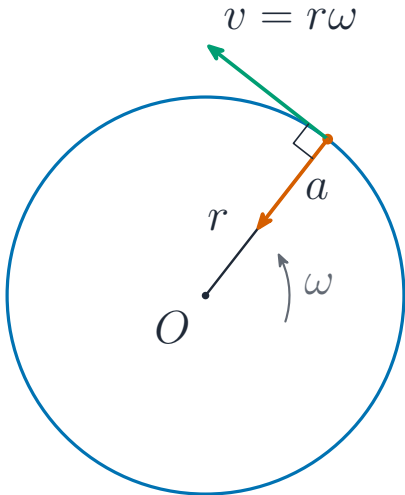
three forces

if a rigid body is in equilibrium under exactly three non-parallel forces, their lines of action are **concurrent**

Either condition alone is not equilibrium. A body can have zero net force and still spin – that is a **couple**, and it is a favourite exam trap.



Circular motion



$v = r\omega$; the **centripetal acceleration** points to the centre:

$$a = r\omega^2 = \frac{v^2}{r}.$$

$$r = 2, \omega = 3 \text{ rad/s}$$

$$v = 6 \text{ m/s}, a = 18 \text{ m/s}^2$$

The hammer thrower is a moving diagram



The hammer thrower is a moving diagram of this page. The taut wire pulls the ball toward the centre – that inward pull is the centripetal force – while the ball's velocity points along the t

Horizontal circles, vertical circles

Angular speed

Angular speed ω links to the speed by $v = r\omega$, and the centripetal acceleration is $a = r\omega^2 = v^2/r$.

Horizontal circle

In a **horizontal circle** the speed is **constant** – as in a **conical pendulum**, a mass swung on a string that traces a cone.

Vertical circle

In a **vertical circle** the speed changes with height, so use **energy conservation**; the **normal contact force** or the string tension supplies the centripetal force.

Where it breaks

The string goes slack (or contact is lost) the moment the required centripetal force exceeds what tension or contact can provide.

Constant speed $\Rightarrow$ Newton's second law alone. Changing speed $\Rightarrow$ energy first, then Newton's second law at the point asked about.

Hooke's law and elastic energy

$$T = \frac{\lambda x}{L}, \quad E = \frac{\lambda x^2}{2L}.$$

λ is the **modulus of elasticity** .

$$L = 2, \lambda = 50, x = 0.5 \text{ m}$$

$$T = 12.5 \text{ N}, E = 3.125 \text{ J}$$

Elastic potential energy

The stored energy

Stretching stores **elastic potential energy** $E = \frac{\lambda x^2}{2L}$, where L is the natural length and λ the **modulus of elasticity**.

Why the square

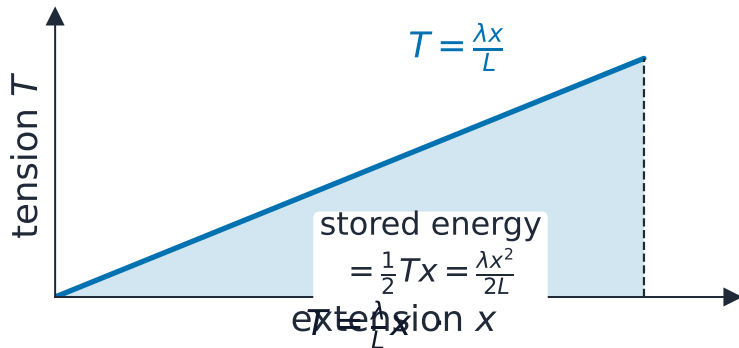
The force grows with the extension, so the energy is the *area* under the force–extension line – hence x^2 , not x .

In an energy equation

It joins kinetic and gravitational potential energy in a conservation statement; the string doing negative work *is* the energy it stores.

A stretched string is a battery: whatever went in comes back out when it returns to its natural length.

Hooke's law and elastic energy, in detail



$$E = \frac{1}{2}Tx = \frac{\lambda x^2}{2L} \quad (\text{area under graph})$$

$$L = 2, \lambda = 50, x = 0.5 \text{ m} \quad T = 12.5 \text{ N}, E = 3.125 \text{ J}$$

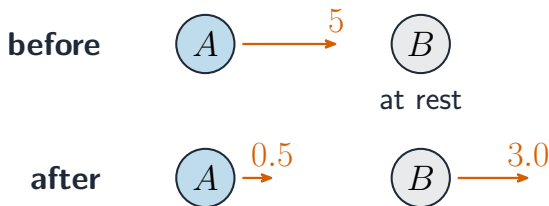


Variable force and momentum

A position-dependent **variable force** uses

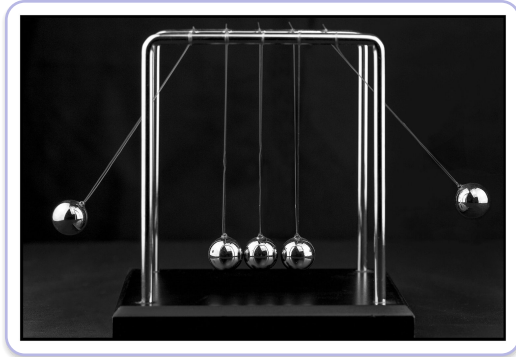
$$a = v \frac{dv}{dx},$$

turning Newton's law into a differential equation in v and x .



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

Momentum



A Newton's cradle demonstrates conservation of momentum in collisions.

Worked example – a collision with restitution

Sphere A (2 kg, 5 m s^{-1}) hits stationary B (3 kg), with $e = 0.5$. Find both final speeds.

- **momentum** : $2(5) = 2v_A + 3v_B$, so
 $10 = 2v_A + 3v_B$
- **restitution** : $v_B - v_A = e(5 - 0) = 2.5$
- **substituting**: $10 = 2v_A + 3(v_A + 2.5)$, so
 $v_A = 0.5 \text{ m s}^{-1}$
- $v_B = 3.0 \text{ m s}^{-1}$

Two equations, always the same two: **conservation of linear momentum** and Newton's law of restitution,
$$e = \frac{\text{separation speed}}{\text{approach speed}}.$$
 Check $v_B > v_A$, or the spheres would still be colliding.

How bouncy is the collision?

Newton's experimental law

Newton's experimental law defines the **coefficient of restitution** : $e = \frac{\text{speed of separation}}{\text{speed of approach}}$, with $0 \leq e \leq 1$.

The two ends

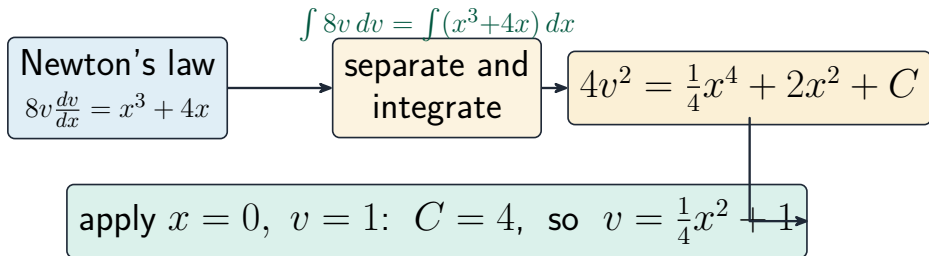
$e = 1$ is perfectly elastic – no kinetic energy lost. $e = 0$ is inelastic: the bodies stay together.

Oblique impact

In an **oblique impact**, resolve the velocities **along** and **perpendicular** to the line of impact. Restitution applies along it; the perpendicular component is unchanged.

Momentum is conserved in every collision; restitution is the second equation you need, and only it tells you about the energy.

Linear motion under a variable force



a position-dependent force becomes a **differential equation** in v and x

A position-dependent force becomes a differential equation in v and x

Use dv/dt when the force depends on time

which form of acceleration?

force depends on **time** t

$$a = \frac{dv}{dt}$$

gives v as a function of t

force depends on **position** x

$$a = v \frac{dv}{dx}$$

gives v as a function of x

both come from $a = \frac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

Use dv/dt when the force depends on time, $v dv/dx$ when it depends on position

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v \frac{dv}{dx}$ and integrate; elastic PE = $\frac{\lambda x^2}{2L}$.



Topic 3 – key takeaways

1

Projectiles

range $u^2 \sin 2\alpha / g$; parabolic path

2

Circular

$v = r\omega$, $a = v^2/r$ toward the centre

3

Hooke

$$T = \lambda x / L; E = \lambda x^2 / 2L$$

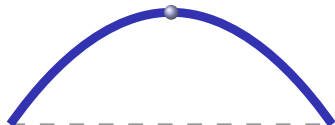
4

Restitution

$e = \text{separation / approach speed}$

Check yourself

- 1 Direction of the centripetal acceleration?
- 2 Tension formula for an elastic string?
- 3 What is e for a perfectly elastic collision?
- 4 Which form of a suits a force depending on x ?



Answers 1. toward the centre 2. $T = \lambda x/L$ 3. $e = 1$ 4. $a = v dv/dx$