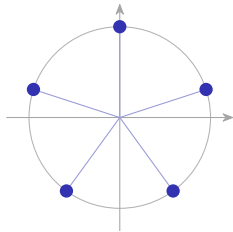


Further Pure Mathematics 2

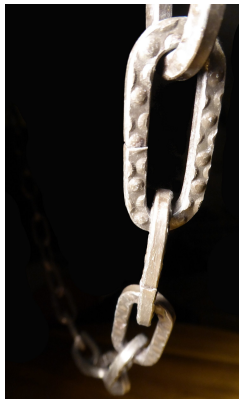
A-Level Further Mathematics ■ Topic 2

A-Level Further Mathematics



What we will cover

- 1 Hyperbolic functions
- 2 Eigenvalues & eigenvectors
- 3 Calculus & Maclaurin series
- 4 Complex numbers
- 5 Differential equations
- 6 Recap



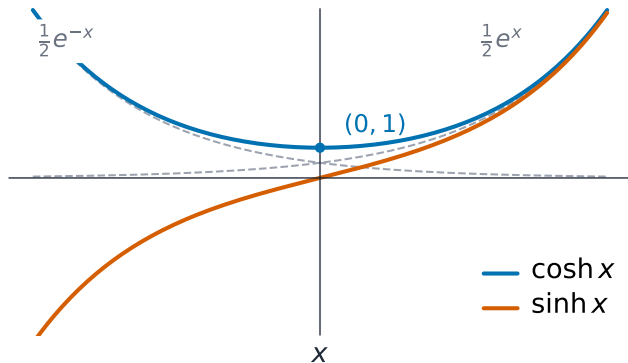


Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}$$

Identities echo trig, with sign changes: $\cosh^2 x - \sinh^2 x = 1$. $(\sinh x)' = \cosh x$,
 $(\cosh x)' = \sinh x$.

Hyperbolic functions, in detail



$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \cdot \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cdot$$
$$\cosh^2 x - \sinh^2 x = 1$$

$\cosh$ is the average of e^x and e^{-x} ; $\sinh$ is half their difference.

The hyperbolic identities, and the logarithmic forms

the main identity

$\cosh^2 x - \sinh^2 x = 1$ – note the **minus**, where the trig version has a plus

the reciprocals

$\operatorname{sech} x = 1/\cosh x$, $\operatorname{cosech} x = 1/\sinh x$, $\coth x = 1/\tanh x$

derived

$1 - \tanh^2 x = \operatorname{sech}^2 x$, and $\coth^2 x - 1 = \operatorname{cosech}^2 x$

logarithmic form

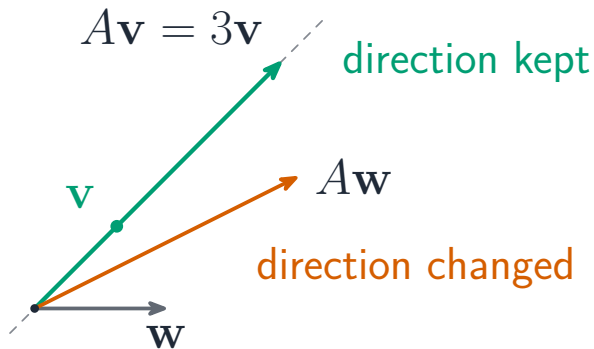
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$

and

$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ for $|x| < 1$

Osborn's rule: take a trig identity and flip the sign of every product of two sines. That generates the hyperbolic identity you need.

Eigenvalues and eigenvectors



Characteristic equation

$\det(A - \lambda I) = 0$ gives the **eigenvalues** ; $Av = \lambda v$ the **eigenvectors** .

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(2 - \lambda)^2 - 1 = 0 \Rightarrow \lambda = 1, 3$$

Then $A = QDQ^{-1}$.

Solving three equations as one matrix equation

the matrix equation

three linear equations in three unknowns become $Ax = b$

non-singular A

$\det A \neq 0$, so $x = A^{-1}b$ – **exactly one** solution

singular A

$\det A = 0$: either **no** solution, or infinitely many. The geometry is three planes with no common point, or a common line

diagonalisation

$A = PDP^{-1}$ with D **diagonal** of eigenvalues and P their eigenvectors

why it helps

$A^n = PD^nP^{-1}$, and raising a diagonal matrix to a power is done entry by entry

Check $\det A$ **first**. It decides whether you invert, or whether you must describe the family of solutions instead.

Worked example – finding eigenvalues

Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- solve $\det(A - \lambda I) = 0$
- $\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 = 0$
- $\lambda^2 - 4\lambda + 3 = 0$, so $\lambda = 1$ or 3

Two checks, both free:
the eigenvalues sum to the **trace** ($1 + 3 = 4$) and multiply to the **determinant** ($1 \times 3 = 3$). Then substitute each λ back to get its eigenvector.



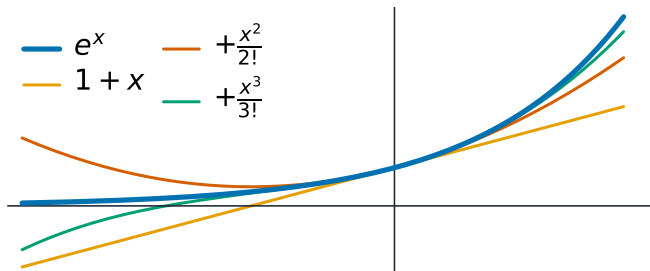
Differentiation and Maclaurin series

Maclaurin series :

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

e.g. $e^x = 1 + x + \frac{x^2}{2!} + \dots$ – more terms, wider match.

Differentiation and Maclaurin series, in detail



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \cdot \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Integration

Standard integrals:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C.$$

$$\int \frac{dx}{\sqrt{4-x^2}}$$

$$= \sin^{-1} \frac{x}{2} + C$$

A **reduction formula** links I_n to I_{n-1} .

Bounding a sum by an integral

the picture

draw rectangles of unit width under, and over, a decreasing curve such as $y = 1/x$

each rectangle

has height $\frac{1}{r}$ and width 1, so its area *is* the term $\frac{1}{r}$

under-estimate

rectangles **below** the curve give $\sum_{r=2}^n \frac{1}{r} < \int_1^n \frac{dx}{x}$

over-estimate

rectangles **above** it give $\int_1^{n+1} \frac{dx}{x} < \sum_{r=1}^n \frac{1}{r}$

the conclusion

the sum sits between two logarithms, so the harmonic series **diverges** – slowly, like $\ln n$

Sketch the rectangles before writing any inequality. **Which side of the curve they sit on** decides the direction of the sign, and that is the whole method.

The standard integrals to know by sight

inverse tan

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

inverse sin

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

inverse sinh

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sinh^{-1} \frac{x}{a} + C$$

inverse cosh

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \frac{x}{a} + C$$

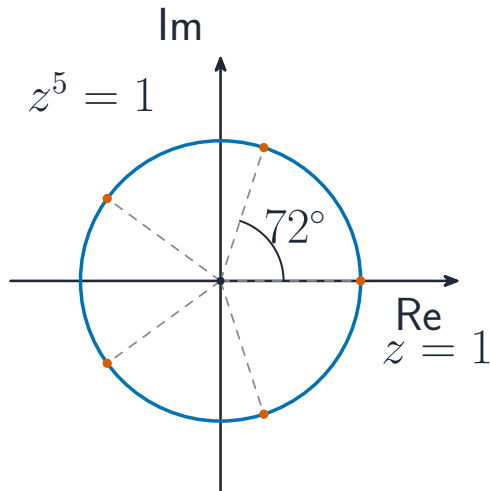
arc length

$$\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \text{ and } \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \text{ in polars}$$

The sign inside the surd chooses the family: $a^2 - x^2$ gives a trig inverse, $a^2 + x^2$ or $x^2 - a^2$ a hyperbolic one. **Complete the square first** when the quadratic is not already in that shape.



Complex numbers – de Moivre



De Moivre :

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Gives multiple-angle formulae and the n **roots of unity** (equally spaced).

Four techniques from the calculus half

Maclaurin's series

Maclaurin's series : $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$ – a Taylor series about the origin.

Separable variables

A differential equation with **separable variables** is integrated after gathering each variable on its own side; one condition fixes the **particular solution**.

Riemann sum

A **Riemann sum** bounds a sum by an integral – rectangles above and below the curve give the two bounds.

Surface of revolution

The **surface area of revolution** is $2\pi \int y \sqrt{1 + (dy/dx)^2} dx$ – the arc length element carried round the axis.

Completing the square is the step that turns an unfamiliar integrand into a standard arctan or arcsin form.

Differential equations

First-order linear: multiply by the **integrating factor** $\mu = e^{\int P dx}$.

$$\frac{dy}{dx} + 2y = e^x$$

$$\mu = e^{2x} \Rightarrow y = \frac{1}{3}e^x + Ce^{-2x}$$

Constant-coefficient: **complementary function** + **particular integral** .

The structure of a linear differential equation's solution

the general solution is the **sum** of two parts

complementary function the solution of the equation with the right side set to **zero**

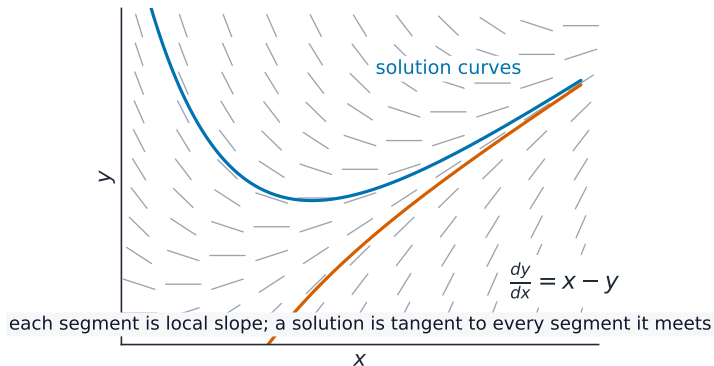
particular integral any single solution of the **full** equation – guessed from the form of the right side

auxiliary equation replace $\frac{d^2y}{dx^2} \rightarrow m^2$, $\frac{dy}{dx} \rightarrow m$, $y \rightarrow 1$; its roots give the complementary function

the three cases distinct real roots $\rightarrow Ae^{m_1x} + Be^{m_2x}$; a **repeated** root $\rightarrow (A + Bx)e^{mx}$; complex $\rightarrow e^{px}(A \cos qx + B \sin qx)$

Find the complementary function first, then the particular integral, then apply the boundary conditions to **the sum** – never to one part alone.

Differential equations, in detail



First-order linear: multiply by the **integrating factor** $\mu = e^{\int P dx}$.

Worked example – an integrating factor, and a repeated root

(a) Solve $\frac{dy}{dx} + 2y = e^x$. **(b) Solve**
 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

- (a) integrating factor $e^{\int 2 dx} = e^{2x}$; so
 $\frac{d}{dx}(ye^{2x}) = e^{3x}$
- $ye^{2x} = \frac{1}{3}e^{3x} + C$, giving $y = \frac{1}{3}e^x + Ce^{-2x}$
- (b) auxiliary: $m^2 - 4m + 4 = (m - 2)^2 = 0$,
 a **repeated** root $m = 2$
- $y = (A + Bx)e^{2x}$

A repeated root needs the extra factor of x – without it the two terms are the same function and you have only one arbitrary constant for a second-order equation.

Complex numbers



Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.



Topic 2 – key takeaways

1

Hyperbolic

$$\cosh^2 - \sinh^2 = 1$$

2

Eigenvalues

$$\det(A - \lambda I) = 0; A = QDQ^{-1}$$

3

Complex

de Moivre; roots of unity

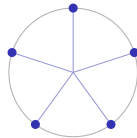
4

DEs

integrating factor; CF + PI

Check yourself

- 1 Simplify $\cosh^2 x - \sinh^2 x$.
- 2 What equation gives the eigenvalues?
- 3 State de Moivre's theorem.
- 4 What multiplies a first-order linear DE?



Answers 1. 1 2. $\det(A - \lambda I) = 0$ 3. $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ 4. the integrating factor