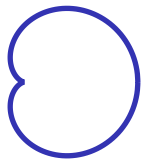


Further Pure Mathematics 1

A-Level Further Mathematics ■ Topic 1

A-Level Further Mathematics



What we will cover

- 1 Roots & series
- 2 Rational functions
- 3 Matrices
- 4 Polar coordinates
- 5 Vectors & induction
- 6 Recap





Roots and summation of series

Roots $\leftrightarrow$ coefficients (quadratic):

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Standard sums: $\sum r = \frac{1}{2}n(n+1)$,
 $\sum r^2 = \frac{1}{6}n(n+1)(2n+1).$

$$\sum_{r=1}^n (2r+1)$$

$$= 2 \cdot \frac{1}{2}n(n+1) + n = n(n+2)$$

Method of differences cancels middle terms.

Roots and coefficients, without solving anything

quadratic

for $ax^2 + bx + c = 0$ with roots α, β : $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

cubic

for $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = -\frac{d}{a}$

the useful identity

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ – and the cubic analogue $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

discriminant

$b^2 - 4ac$ decides how many real roots there are before you find any of them

the point

symmetric functions of the roots come straight from the **coefficients** – **the roots themselves are never needed**

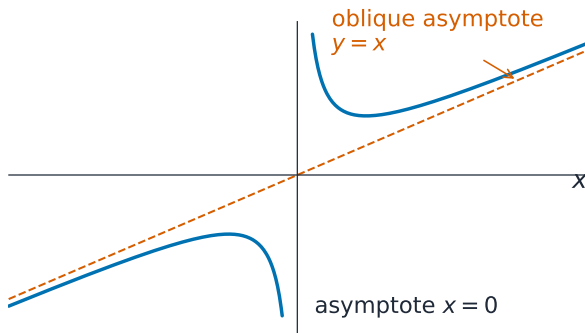
When a question asks for $\alpha^2 + \beta^2$ or $\frac{1}{\alpha} + \frac{1}{\beta}$, do not solve the equation. Build it from the two sums.

Rational functions

Vertical asymptotes: denominator zero (numerator not). Horizontal: compare degrees.

If the numerator's degree is one more than the denominator, divide first – the quotient is an **oblique** asymptote.

Rational functions, in detail



oblique asymptote when $\deg(\text{num}) = \deg(\text{den}) + 1$
(long division)

Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertic

Sketching a rational function

rational function

a fraction of two polynomials

vertical asymptotes

where the denominator is zero (and the numerator is not)

horizontal

when the degrees are equal, y tends to the ratio of the leading coefficients

oblique asymptote

when the top's degree is **exactly one higher**: divide out to get $y = mx + c$ plus a vanishing remainder

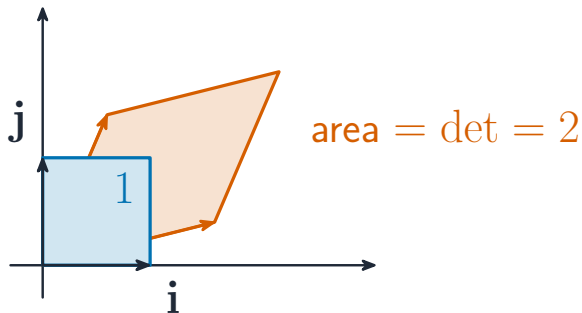
the checks

axis intercepts, sign of the function between asymptotes, and any stationary points

The oblique case is the one the exam sets, and **polynomial division** is how you find it. Divide first, sketch second.



Matrices



$\det A = ad - bc$ (the area scale factor).

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

A 2×2 matrix is a **transformation** ; AB means “do B then A ”.

The matrix operations, and the transformations they are

addition

entry by entry, in matching positions – so the shapes must match

multiplication

row into column; AB needs A 's columns to equal B 's rows, and $AB \neq BA$ in general

identity matrix

I , with ones on the diagonal: $AI = IA = A$

as a geometric transformation

a 2×2 matrix maps the plane – rotation, reflection, **enlargement**, shear, or a composition

the determinant

$\det A$ is the **area scale factor**; $\det A = 0$ means the transformation collapses the plane onto a line

invariant line

a line the transformation maps onto itself – found from $A\mathbf{v} = \lambda\mathbf{v}$

Composing transformations is multiplying matrices, **right to left**: BA means do A first. Getting that order wrong is the standard lost mark.

Matrices: the special ones, and what they do to a plane

Zero matrix

The **zero matrix** sends every vector to the origin – and multiplying by it destroys all information.

Inverse matrix

The **inverse matrix** undoes the transformation: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, and it exists only when $\det \mathbf{A} \neq 0$.

Singular matrix

A **singular matrix** has zero determinant: it collapses the plane onto a line, so it cannot be undone.

Invariant points

Invariant points satisfy $\mathbf{A}\mathbf{x} = \mathbf{x}$ – fixed by the transformation. A **stretch** fixes the axis it stretches along.

The determinant is the area scale factor. Zero means the area is destroyed, which is exactly why the inverse fails.



Polar coordinates

Polar (r, θ) : $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$.

$$\text{area} = \frac{1}{2} \int r^2 d\theta.$$

e.g. $r = 4 \cos \theta \Rightarrow (x - 2)^2 + y^2 = 4$.

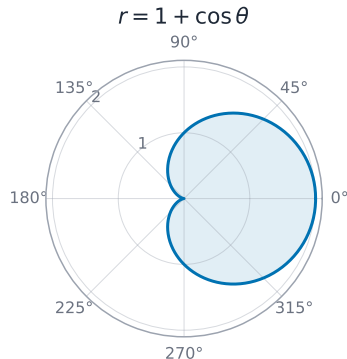
Worked example – polar to Cartesian

Convert $r = 4 \cos \theta$ to Cartesian coordinates and identify the curve.

- multiply both sides by r : $r^2 = 4r \cos \theta$
- use $r^2 = x^2 + y^2$ and $r \cos \theta = x$:
 $x^2 + y^2 = 4x$
- complete the square: $(x - 2)^2 + y^2 = 4$
- a circle of radius 2 centred at $(2, 0)$

Multiplying by r is the move. It turns $r \cos \theta$ into x and r^2 into $x^2 + y^2$, and nothing else does.

Polar coordinates, in detail

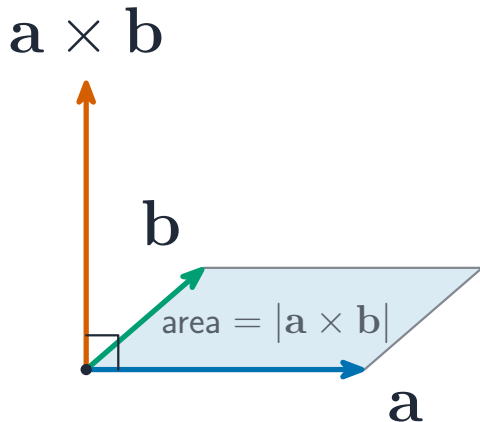


cardioid · polar area $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

the cardioid $r = 1 + \cos \theta$

The cardioid $r = 1 + \cos \theta$, plotted straight from r as a function of θ .

Vectors



A **plane** : $\mathbf{r} \cdot \mathbf{n} = p$. The **vector product**

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}}$$

is perpendicular to both; its length is the parallelogram's area.

Three ways to write a plane

Cartesian

$ax + by + cz = d$, where (a, b, c) is a **normal** to the plane

scalar product

$\mathbf{r} \cdot \mathbf{n} = p$ – the same statement, with the normal made explicit

parametric

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$: a point plus two independent directions in the plane

converting

the cross product $\mathbf{b} \times \mathbf{c}$ gives the normal, which turns the parametric form into the other two

Choose the form the question wants: **Cartesian** for a distance from a point, **parametric** for an intersection with a line.

Proof by induction

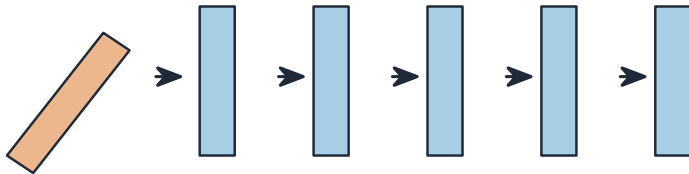
Induction :

1 base case $n = 1$

2 inductive step: true for $k \Rightarrow$ true for $k + 1$

Then it holds for every n .

so every domino falls — true for all n



Induction, series, and the vocabulary around them

Mathematical induction

Mathematical induction : prove the base case, assume for $n = k$, then prove for $n = k + 1$. An **inductive proof** must state all three steps.

Sum to infinity

A geometric series is **convergent** when $|r| < 1$, and its **sum to infinity** is $\frac{a}{1 - r}$.

Partial fractions

Partial fractions split a rational function into simpler pieces – which is what makes a telescoping sum or an integral doable.

Substitution and curves

A **substitution** converts an integral into a standard form; **polar curves** $r = f(\theta)$ have area $\frac{1}{2} \int r^2 d\theta$; **skew lines** in 3D are neither parallel nor intersecting.

The induction step must *use* the assumption. A proof that reaches $n = k + 1$ without it has proved nothing.

Induction, and the sentence each step must contain

base case

show the statement holds for the first n – usually $n = 1$

inductive step

assume it holds for $n = k$, then **prove** it for $n = k + 1$

the conclusion

“true for $n = 1$, and true for $k + 1$ whenever true for k , so true for all $n \geq 1$ by induction”

conjecture

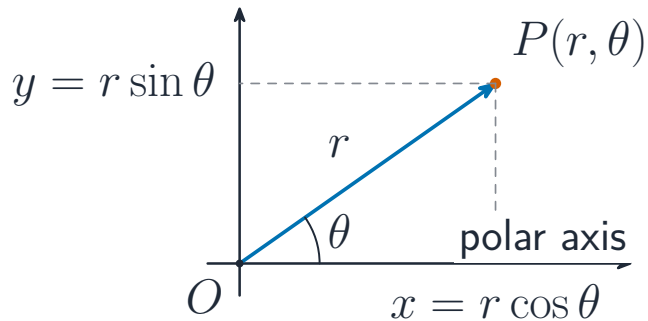
often you compute a few cases first, **guess** the pattern, and only then confirm it by **induction**

where marks go

stating the assumption explicitly, and writing the conclusion sentence in full

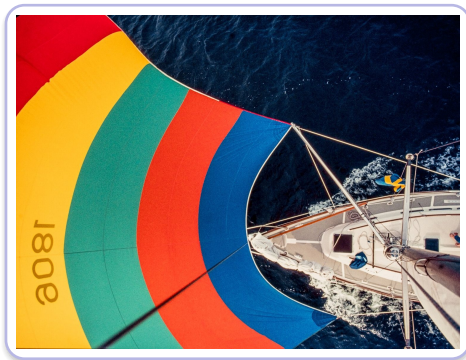
The proof is not finished when the algebra closes. The **final sentence** is a marked step, and omitting it is the commonest way to lose a mark on an otherwise correct proof.

Polar coordinates, continued



Polar coordinates (r, θ) convert to Cartesian by $x = r \cos \theta$ and $y = r \sin \theta$.

Vectors, continued



Forces like wind and water are vectors – they have both size and direction.

Worked example – a cross product

Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

$$\blacksquare \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\blacksquare = \mathbf{i}(1 \cdot 1 - 0 \cdot 1) - \mathbf{j}(1 \cdot 1 - 0 \cdot 0) + \mathbf{k}(1 \cdot 1 - 1 \cdot 0)$$

$$\blacksquare = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Check it: the result is **perpendicular to both**, so $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ must be zero. It is. That check costs seconds and catches every sign slip.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k + 1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.



Topic 1 – key takeaways

1

Roots

sums & products link to coefficients

2

Matrices

$\det = ad - bc$; transformations

3

Polar

$x = r \cos \theta$; area $\frac{1}{2} \int r^2 d\theta$

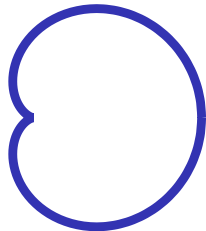
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Induction

base case + inductive step

Check yourself

- 1 For $ax^2 + bx + c = 0$, what is $\alpha\beta$?
- 2 What is $\det \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$?
- 3 Convert (r, θ) to the x -coordinate.
- 4 The two steps of proof by induction?



Answers 1. c/a 2. 10 3. $x = r \cos \theta$ 4. base case, then inductive step