

Further Probability & Statistics

A-Level Further Mathematics

This handout covers Topic 4: **Further Probability & Statistics**.

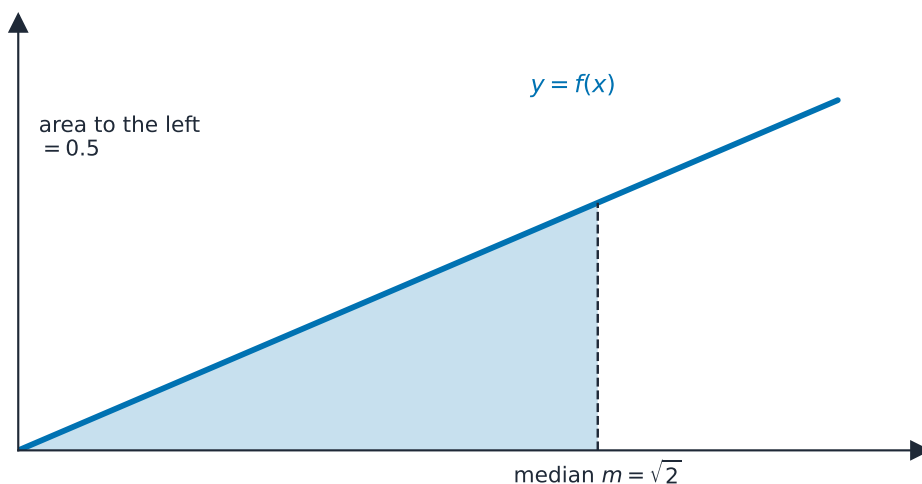
It adds continuous distributions, small-sample inference, the chi-squared and non-parametric tests, and probability generating functions.

Continuous random variables

A continuous variable X is described by a **probability density function**

$f(x)$, which may be defined piecewise. The probability over a range is the area under f , and the mean of any function of X is

$$E(g(X)) = \int g(x) f(x) dx.$$



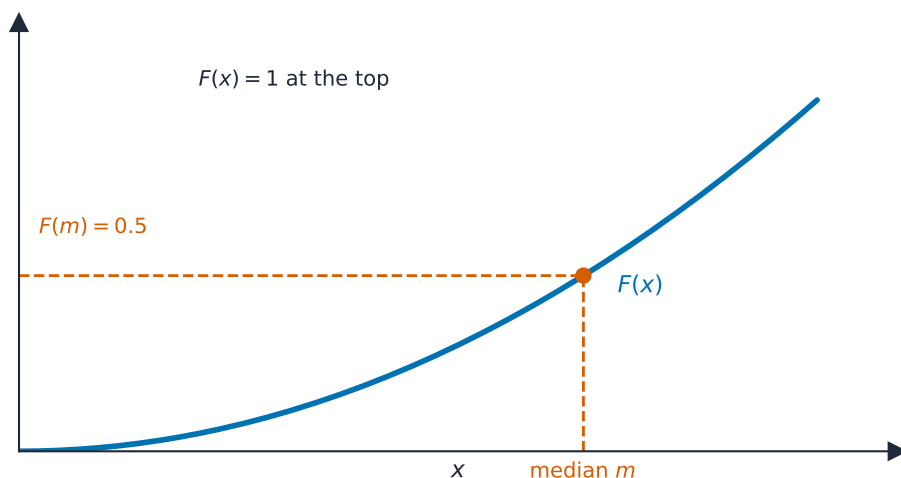
$$\text{median } m \text{ solves } F(m) = \int_{-\infty}^m f(x) dx = \frac{1}{2}$$

The median sits where the area under $f(x)$ to its left is exactly 0.5.

The **cumulative distribution function**

$F(x) = P(X \leq x)$ is the

running total: $F(x) = \int_{-\infty}^x f(t) dt$, and $f(x) = F'(x)$. Use F to find probabilities and percentiles (for example, the **median** solves $F(x) = 0.5$).



$$F(x) = P(X \leq x) = \int_{-\infty}^x f \cdot 0 \leq F \leq 1 \cdot \text{median: } F(m) = \frac{1}{2}$$

The cumulative graph $F(x)$ rises from 0 to 1; the height 0.5 is reached at the median.

Worked example. A variable has $f(x) = \frac{1}{2}x$ for $0 \leq x \leq 2$. Find the median.

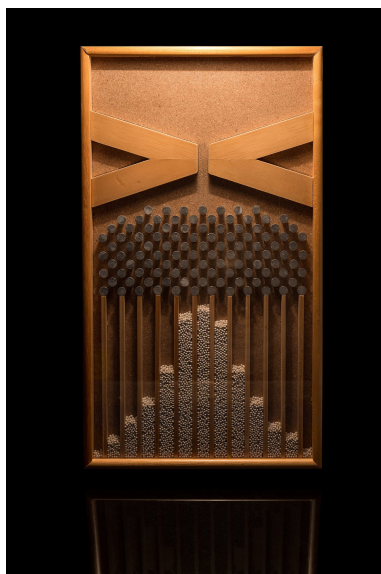
The cumulative distribution function is $F(x) = \int_0^x \frac{1}{2}t \, dt = \frac{1}{4}x^2$. Set $F(m) = 0.5$:

$$\frac{1}{4}m^2 = 0.5 \Rightarrow m^2 = 2 \Rightarrow m = \sqrt{2} = 1.41.$$

The distribution of a related variable. If $Y = g(X)$, find Y 's distribution through its cumulative function. For $Y = X^2$: $F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$, then differentiate for $f_Y = F_Y'$. When g is monotonic increasing there is a shortcut, $F_Y(y) = F_X(g^{-1}(y))$.

Worked example. With $f_X(x) = \frac{1}{2}x$ on $[0, 2]$ (so $F_X(x) = \frac{1}{4}x^2$), let $Y = X^2$. For $0 \leq y \leq 4$, $F_Y(y) = F_X(\sqrt{y}) = \frac{1}{4}y$, so $f_Y(y) = F_Y'(y) = \frac{1}{4}$ – that is, Y is uniform on $[0, 4]$.

Inference using normal and t-distributions

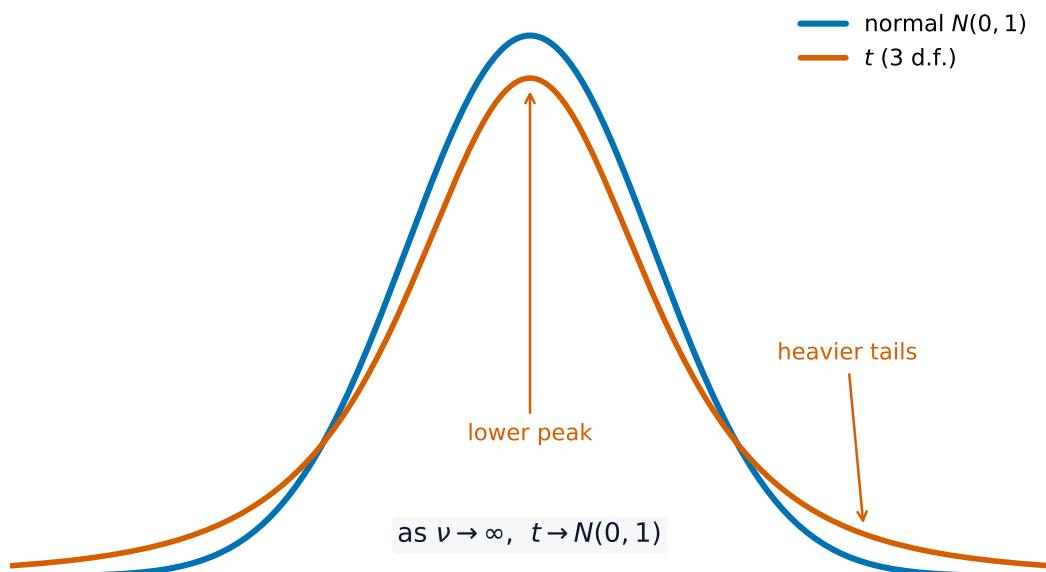


A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

When a sample is small and the population variance is unknown, base your **hypothesis test** on the t -distribution instead of the normal. The same idea gives a **confidence interval** for the mean:

$$\bar{x} \pm t \frac{s}{\sqrt{n}},$$

where t comes from the t -tables with $n - 1$ degrees of freedom. To compare two populations, use a **two-sample** (2-sample) or **paired-sample** t -test, after finding a **pooled estimate** of the shared variance when appropriate.



For a small sample the t -distribution is flatter with heavier tails, so its critical values are larger.

Worked example. A sample of $n = 10$ has mean $\bar{x} = 50$ and standard deviation $s = 4$. Find a 95% confidence interval for the mean (use $t = 2.262$ for 9 degrees of freedom).

$$50 \pm 2.262 \times \frac{4}{\sqrt{10}} = 50 \pm 2.86 \Rightarrow (47.1, 52.9).$$

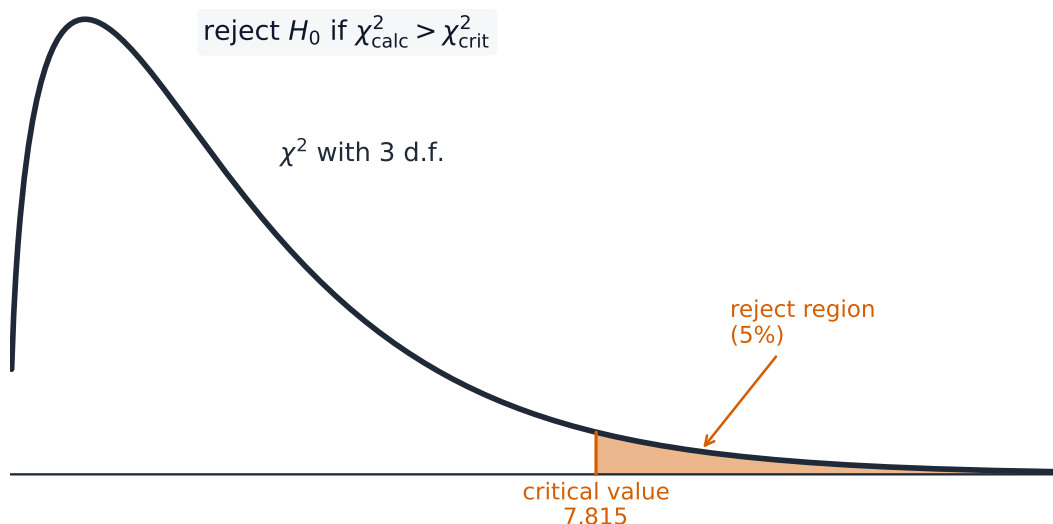
Chi-squared tests

A χ^2 -test (**chi-squared test**) compares observed counts O with expected counts E from a **theoretical distribution**:

$$\chi^2 = \sum \frac{(O - E)^2}{E}.$$

Compare this with a table value for the right number of **degrees of freedom**.

Two uses: a **goodness of fit** test (does the data follow the proposed model?), and a test for **independence** of two variables in a **contingency table**.



The test rejects the model when χ^2 exceeds the critical value, landing in the shaded 5% tail.

Worked example. Four equally likely categories give observed counts 20, 30, 25, 25 (so each expected count is 25). Test the fit at the 5% level.

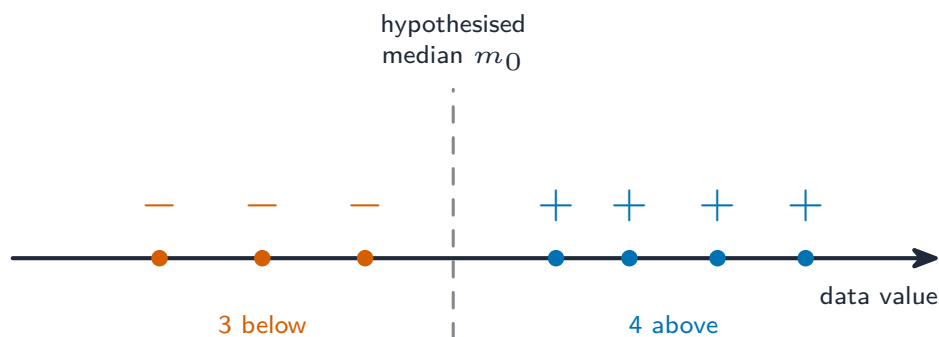
$$\chi^2 = \frac{(20 - 25)^2 + (30 - 25)^2 + 0 + 0}{25} = \frac{25 + 25}{25} = 2.$$

With $4 - 1 = 3$ degrees of freedom the table value is 7.815. Since $2 < 7.815$, do not reject the model.

Non-parametric tests

A **non-parametric test** makes no assumption that the data is normal, so it is useful when that assumption fails. The basic ones are:

- the **sign test** : count how many values fall above and below a proposed median, and test those counts with a binomial model;
- the **Wilcoxon signed-rank test** (the **matched-pairs** test for paired data), which also uses the sizes of the differences, not just their signs;
- the **Wilcoxon rank-sum test** , for comparing two separate samples.



The sign test counts how many values fall above and below the proposed median, then tests those counts with a binomial model

Worked example. Test whether a median is 5. In a sample of 10 values (none equal to 5), 9 lie **above** 5 and 1 lies below. Test at the 5% level (two-tailed).

Under H_0 (median = 5) the number above follows $B(10, 0.5)$. The observed result (9 above) is extreme, so find $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + (0.5)^{10} = \frac{11}{1024} = 0.0107$. For a two-tailed test compare with $\frac{1}{2}(5\%) = 0.025$. Since $0.0107 < 0.025$, **reject** H_0 : there is evidence the median is not 5.

Probability generating functions



Dice: a starting point for probability and discrete random variables.

The **probability generating function**

of a discrete variable X is

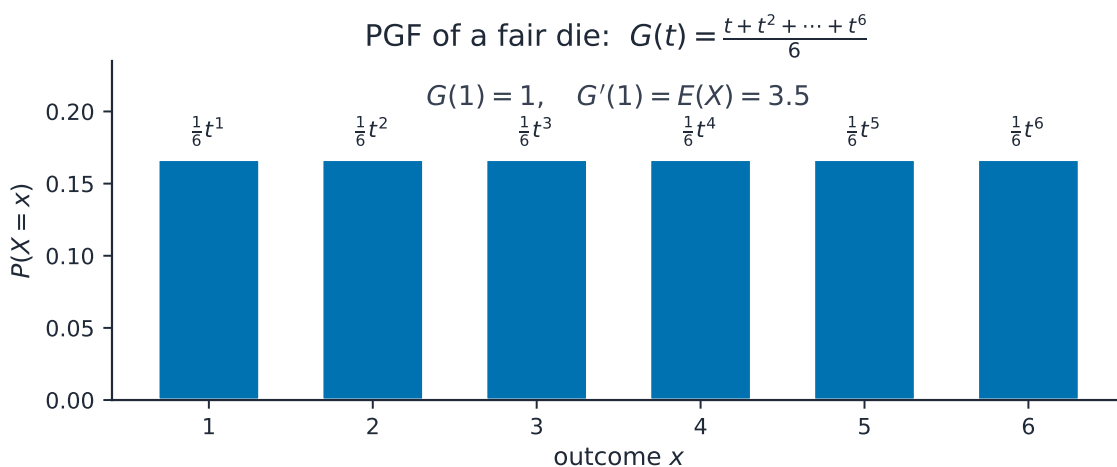
$$G(t) = E(t^X) = \sum_x P(X = x) t^x.$$

It packs the whole distribution into one function. The mean and variance come from its derivatives at $t = 1$: $E(X) = G'(1)$ and $\text{Var}(X) = G''(1) + G'(1) - (G'(1))^2$. Also, the PGF of a sum of independent variables is the product of their PGFs.

Worked example. X has $P(X = 0) = 0.5$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$. Find $E(X)$ using the PGF.

Here $G(t) = 0.5 + 0.3t + 0.2t^2$, so $G'(t) = 0.3 + 0.4t$ and

$$E(X) = G'(1) = 0.3 + 0.4 = 0.7.$$



$$G(t) = E(t^X) \cdot E(X) = G'(1) \cdot \text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$$

PGF of a fair die packs equal masses at 1 through 6 into $G(t)$

Exam tips

- For a **continuous random variable**, the pdf integrates to 1 over its range, and $E(X) = \int xf(x) dx$.
- Use the **t -distribution** when the sample is small and the population variance is unknown; state the degrees of freedom.
- For a **chi-squared** test compute $\sum(O - E)^2/E$, compare with the critical value at the right degrees of freedom, and **combine classes with** $E < 5$.
- State H_0 and H_1 and give the conclusion **in context** for every test.