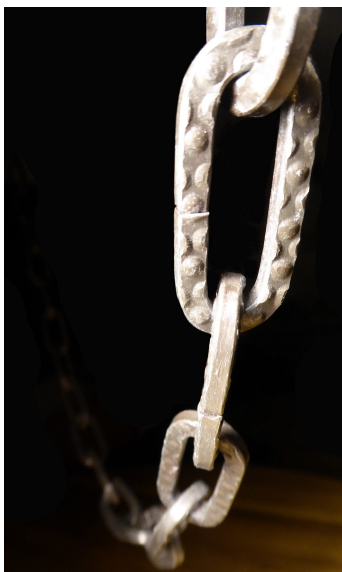


Further Pure Mathematics 2

A-Level Further Mathematics

This handout covers Topic 2: **Further Pure Mathematics** 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

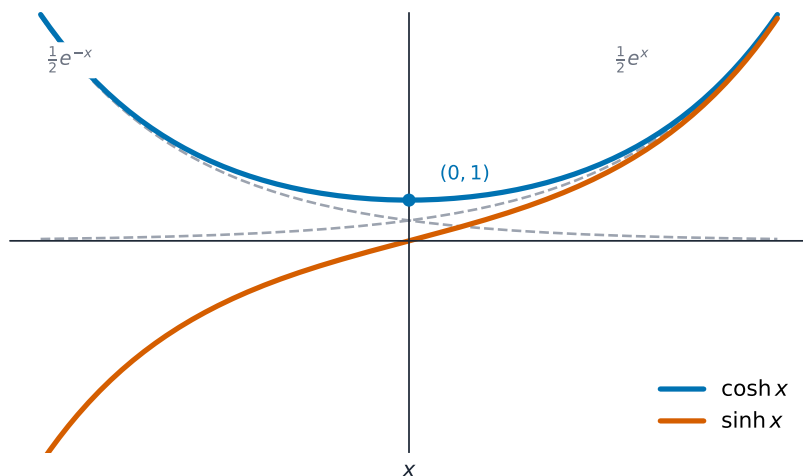
Hyperbolic functions



A hanging cable forms a catenary – the curve of the hyperbolic cosine.

The **hyperbolic functions** are built from the exponential function:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}.$$



$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh^2 x - \sinh^2 x = 1$$

cosh is the average of e^x and e^{-x} ; sinh is half their difference.

They obey identities much like the trigonometric ones, the main one being $\cosh^2 x - \sinh^2 x = 1$. The three **reciprocals** complete the set: $\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, and $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$ – all built from e^x too, and worth recognising in integrals and mark schemes. The **inverse hyperbolic functions** have a **logarithmic form**, for example $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

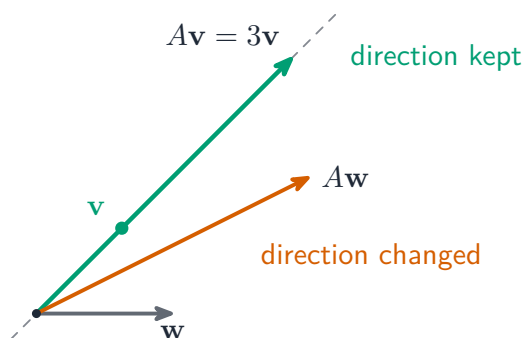
Worked example. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

Matrices

You can write three linear equations in three unknowns as a single **matrix equation** $A\mathbf{x} = \mathbf{b}$. If A is non-singular there is one solution; if A is singular the equations are either inconsistent (no solution) or have infinitely many.

For a square matrix A , the **characteristic equation** is $\det(A - \lambda I) = 0$. Its solutions are the **eigenvalues** λ , and for each one the vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector**. You can then write $A = QDQ^{-1}$, where D is a **diagonal matrix** of eigenvalues and the columns of Q are the eigenvectors.



Multiplying an eigenvector by A only stretches it; a general vector is turned to a new direction.

Worked example. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2 - 1 = 0 \Rightarrow 2 - \lambda = \pm 1.$$

So $\lambda = 1$ or $\lambda = 3$. (The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

Differentiation

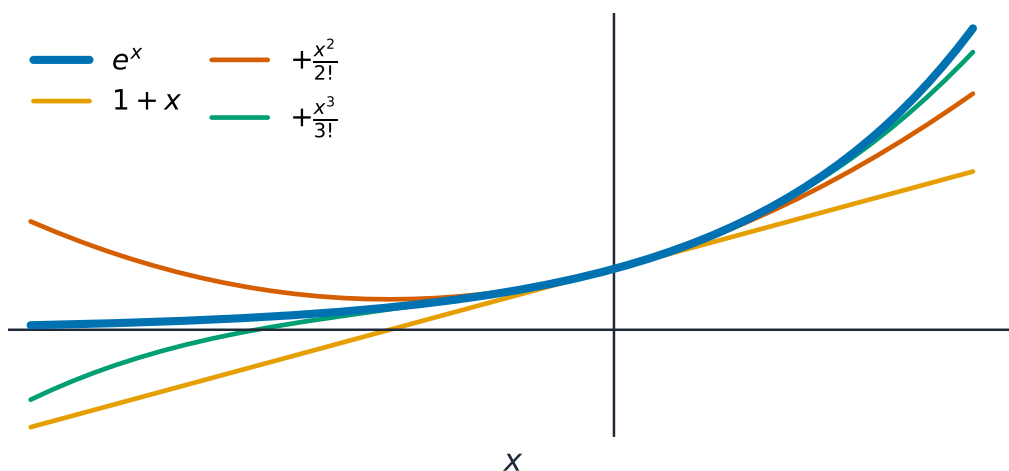
You can now differentiate the hyperbolic functions and the inverse functions $\sin^{-1} x$, $\tan^{-1} x$, $\sinh^{-1} x$ and so on. You can also find $\frac{d^2 y}{dx^2}$ for curves given implicitly or parametrically.

A **Maclaurin's series**

writes a function as a power series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

For example $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \cdot \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Worked example. Find the Maclaurin series of $f(x) = \ln(1+x)$ up to the term in x^3 .

$f(0) = 0$; $f'(x) = \frac{1}{1+x}$ so $f'(0) = 1$; $f''(x) = \frac{-1}{(1+x)^2}$ so $f''(0) = -1$; $f'''(x) = \frac{2}{(1+x)^3}$ so $f'''(0) = 2$. Substituting into the series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Integration

Learn these standard integrals:

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C.$$

A **trigonometric substitution**, a **hyperbolic substitution**, or **completing the square** in the denominator handles related forms. A **reduction formula** links an integral I_n to I_{n-1} , so you can work down step by step. Integration also gives the **arc length** of a curve and the **surface area of revolution** when a curve is turned about an axis.

Worked example. Find $\int \frac{1}{\sqrt{4-x^2}} dx$.

Here $a^2 = 4$, so $a = 2$ and

$$\int \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1} \frac{x}{2} + C.$$

Bounding a sum by an integral. Draw rectangles of unit width under, or over, a decreasing curve such as $y = \frac{1}{x}$. Each rectangle of height $\frac{1}{r}$ is trapped between two integral strips, so summing gives

$$\ln(n+1) < \sum_{r=1}^n \frac{1}{r} < 1 + \ln n.$$

Shrinking the rectangle width to $\frac{1}{n} \rightarrow 0$ turns such a sum into a **Riemann sum**

that equals the integral in the limit, e.g. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx$.

Complex numbers

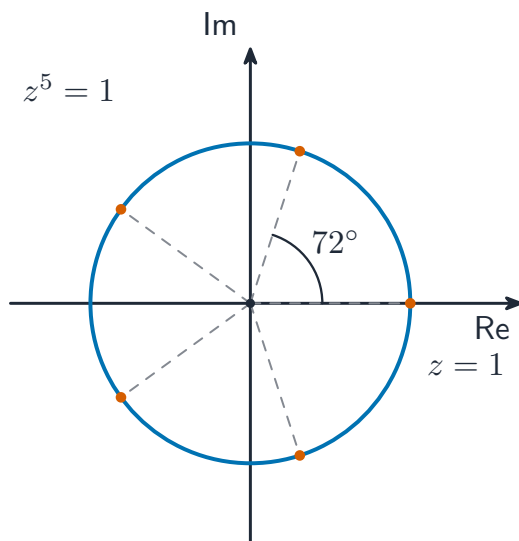


Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

A **complex number** in polar form is $z = r(\cos \theta + i \sin \theta)$. **De Moivre's theorem** says that for any integer n ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

It is used to expand $\cos n\theta$ and $\sin n\theta$ in powers, to sum series, and to find the n **roots of unity** (the solutions of $z^n = 1$, equally spaced around the unit circle).



The five solutions of $z^5 = 1$ are equally spaced 72° apart on the unit circle.

Worked example. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos \theta$.

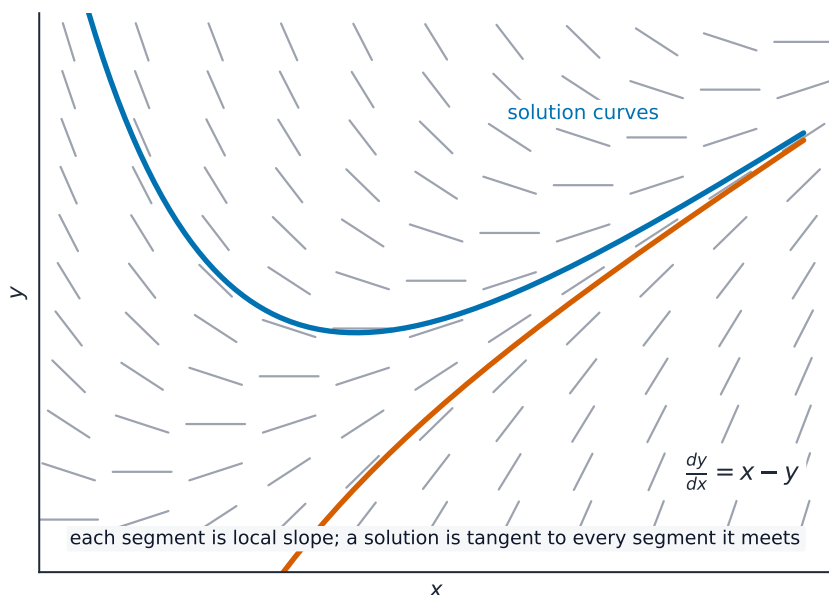
Take the real part of $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$:

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta) = 4 \cos^3 \theta - 3 \cos \theta.$$

Differential equations

For a first order linear equation $\frac{dy}{dx} + P(x)y = Q(x)$, multiply by the **integrating factor** $\mu = e^{\int P dx}$. The left side then becomes $\frac{d}{dx}(\mu y)$, so you can integrate directly.

For a linear equation with constant coefficients, the **general solution** is the sum of two parts: the **complementary function** (the solution of the equation with the right side set to 0, found from the auxiliary equation) and a **particular integral** (any one solution of the full equation). **Initial conditions** then fix the constants to give the **particular solution**; simpler equations with **separable variables** are integrated directly.



A solution curve follows the slope field, staying tangent to the little segments everywhere.

Worked example. Solve $\frac{dy}{dx} + 2y = e^x$.

The integrating factor is $\mu = e^{\int 2 dx} = e^{2x}$. Multiplying through gives $\frac{d}{dx}(y e^{2x}) = e^{3x}$, so

$$y e^{2x} = \frac{1}{3} e^{3x} + C \Rightarrow y = \frac{1}{3} e^x + C e^{-2x}.$$

Second order, constant coefficients. For $\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$, solve the **auxiliary equation** $m^2 + bm + c = 0$. The complementary function depends on its roots:

- two distinct real roots m_1, m_2 : $y = A e^{m_1 x} + B e^{m_2 x}$;
- a repeated root m : $y = (A + Bx) e^{mx}$;
- complex roots $p \pm qi$: $y = e^{px} (A \cos qx + B \sin qx)$.

Then add a particular integral by trying a form matching $f(x)$ (a polynomial, $a e^{bx}$, or $a \cos px + b \sin px$); if that trial already appears in the complementary function, **multiply it by x** .

Worked example. Solve $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$. The auxiliary equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ has a repeated root $m = 2$, so $y = (A + Bx) e^{2x}$.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.