

Further Pure Mathematics 1

A-Level Further Mathematics

This handout covers Topic 1: **Further Pure Mathematics** 1. It adds new algebra, matrices, polar coordinates, more vectors, and proof by induction.

Roots of polynomial equations

For a polynomial equation, the **roots** are linked to the **coefficients**. For $ax^2 + bx + c = 0$ with roots α, β :

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

For a cubic $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ :

$$\sum \alpha = -\frac{b}{a}, \quad \sum \alpha\beta = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

Sums like $\sum \alpha$ and $\sum \alpha\beta$ are **symmetric functions** of the roots (unchanged if the roots are swapped). To find an equation whose roots are changed in a simple way, use a **substitution** (for example, put $w = \alpha + 1$).

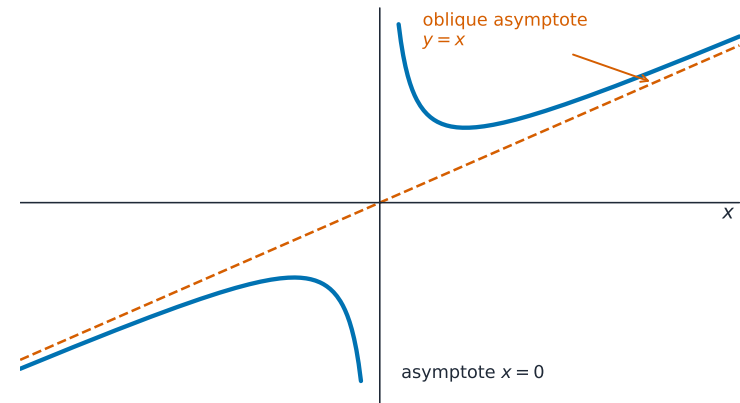
Worked example. The equation $x^2 - 5x + 6 = 0$ has roots α, β . Find the equation with roots $\alpha + 1, \beta + 1$.

Here $\alpha + \beta = 5$ and $\alpha\beta = 6$. The new sum is $(\alpha + 1) + (\beta + 1) = 7$, and the new product is $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 6 + 5 + 1 = 12$. So the new equation is

$$x^2 - 7x + 12 = 0.$$

Rational functions and graphs

A **rational function** is a fraction of two polynomials. When the top has degree exactly one higher than the bottom, the graph has an **oblique asymptote** (a slanted line the curve approaches); find it by dividing out. You should also be able to relate the graph of $y = f(x)$ to those of $y^2 = f(x)$, $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$. Find the set of values the function can take using the **discriminant**, and locate its **turning points**.



oblique asymptote when $\deg(\text{num}) = \deg(\text{den}) + 1$ (long division)

Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertical asymptote.

Worked example. Find the oblique asymptote of $y = \frac{x^2 + x + 1}{x + 1}$.

Divide out the fraction: $\frac{x^2 + x + 1}{x + 1} = x + \frac{1}{x + 1}$. As $x \rightarrow \pm\infty$ the remainder $\frac{1}{x + 1} \rightarrow 0$, so the curve approaches the line $y = x$ – the oblique asymptote.

Summation of series

Learn the standard results:

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1), \quad \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2.$$

The **method of differences** sums a series by cancelling middle terms: if **partial fractions** write each term as $f(r) - f(r+1)$, almost everything cancels. From the sum to n terms you can see whether a series is **convergent** and, if so, find its **sum to infinity**.

Worked example. Find $\sum_{r=1}^n (2r + 1)$.

$$\sum_{r=1}^n (2r + 1) = 2 \sum_{r=1}^n r + \sum_{r=1}^n 1 = 2 \cdot \frac{1}{2}n(n+1) + n = n(n+1) + n = n(n+2).$$

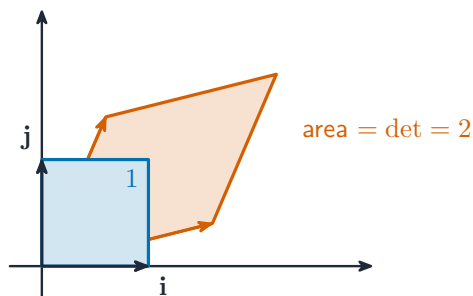
Matrices

A **matrix** is a rectangular block of numbers. **Matrix addition** adds entries in matching positions; you can also subtract and multiply matrices (multiplication is row $\times$ column). The **zero matrix** has every entry 0, and the **identity matrix** (or **unit matrix**) I leaves any matrix unchanged under multiplication.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** is $\det A = ad - bc$. If $\det A \neq 0$ the matrix is non-singular (otherwise it is a **singular matrix**), and the **inverse matrix** is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For a product, $(AB)^{-1} = B^{-1}A^{-1}$. A 2×2 matrix can represent a **geometric transformation** of the plane – a **rotation**, **reflection**, **enlargement**, **stretch** or **shear**: the determinant gives the area scale factor, and a product AB means "do B , then A ". Points or lines that do not move are called **invariant points** and **invariant lines**.



The matrix sends the unit square to a parallelogram whose area is the determinant.

Worked example. Find the inverse of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

$\det A = 3 \times 4 - 1 \times 2 = 10$, so

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

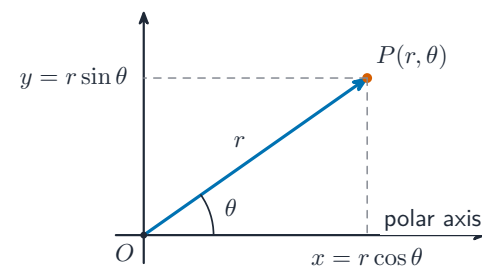
Polar coordinates



A spiral staircase: spirals are described naturally using polar coordinates.

Polar coordinates give a point by its distance r from the origin and its angle θ . They link to **Cartesian coordinates** by

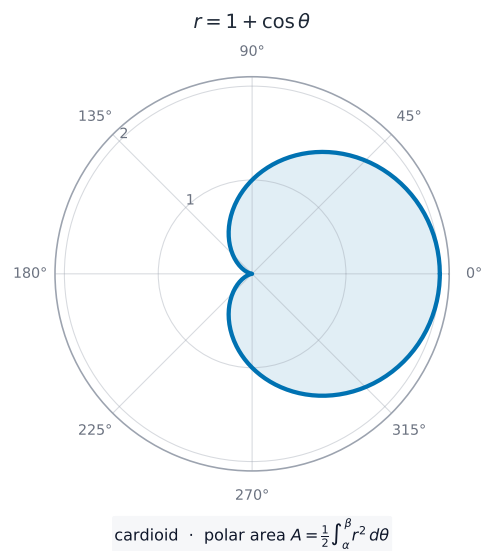
$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$



Polar coordinates (r, θ) convert to Cartesian by $x = r \cos \theta$ and $y = r \sin \theta$.

You should sketch simple **polar curves**, and find the area of a sector with

$$\text{area} = \frac{1}{2} \int r^2 d\theta.$$



The cardioid $r = 1 + \cos \theta$, plotted straight from r as a function of θ .

Worked example. Convert the polar equation $r = 4 \cos \theta$ to Cartesian form.

Multiply both sides by r : $r^2 = 4r \cos \theta$, so $x^2 + y^2 = 4x$. Completing the square gives $(x - 2)^2 + y^2 = 4$, a circle of radius 2 centred at $(2, 0)$.

Vectors



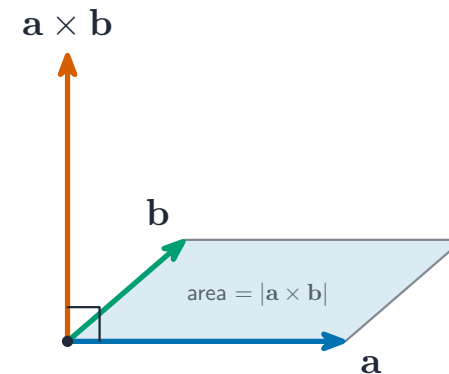
Forces like wind and water are vectors – they have both size and direction.

In three dimensions a **plane** can be written as $ax + by + cz = d$, or $\mathbf{r} \cdot \mathbf{n} = p$, or

$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$. Besides the scalar product, there is the **vector product** of two **vectors** :

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}},$$

which gives a vector at right angles to both. With scalar and vector products you can find distances, angles, and where lines and planes meet – including the shortest distance between **skew lines** .



$\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors; its length equals the area of the parallelogram they span.

Worked example. Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(1) + \mathbf{k}(1) = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

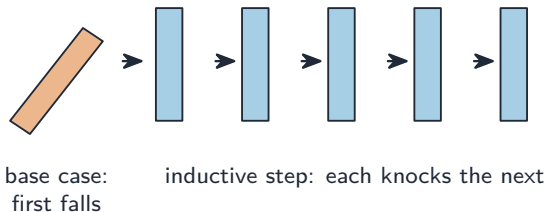
Proof by induction

Mathematical induction proves a result for every positive integer n in two steps:

1. **Base case:** show the result is true for $n = 1$.
2. **Inductive step:** assume it is true for $n = k$, then prove it for $n = k + 1$.

If both steps work, the result is true for all n . Often you first make a **conjecture** (a sensible guess) from a few cases, then confirm it by **inductive proof** .

so every domino falls — true for all n



Induction is like dominoes: the base case topples the first, and the inductive step makes each one knock over the next – so the result holds for every n

Worked example. Prove that $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$.

Base case $n = 1$: the left side is 1 and the right side is $\frac{1}{2}(1)(2) = 1$. True. **Inductive**

step: assume $\sum_{r=1}^k r = \frac{1}{2}k(k+1)$. Then

$$\sum_{r=1}^{k+1} r = \frac{1}{2}k(k+1) + (k+1) = (k+1) \left(\frac{k}{2} + 1 \right) = \frac{1}{2}(k+1)(k+2).$$

This is the formula with $n = k+1$, so by induction it holds for all n .

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k+1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.