

Further Pure Mathematics 1

A-Level Further Mathematics

This handout covers Topic 1: **Further Pure Mathematics** 1. It adds new algebra, matrices, polar coordinates, more vectors, and proof by induction.

Roots of polynomial equations

Before trusting a program's calculated roots, check their sum and product against its stored polynomial **coefficients**. These relations also give useful sums when finding the individual roots would be difficult. A **root** makes the polynomial zero. Count repeated roots with their multiplicities and include complex roots; a real graph shows only the real ones. The relations here apply to degrees 2, 3 and 4, always with a non-zero leading coefficient.

■ Derive the relations from factors

Expand $a(x - \alpha)(x - \beta)$ and compare coefficients with $ax^2 + bx + c$. This gives

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

For $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ , expansion similarly gives

$$e_1 = \alpha + \beta + \gamma = -\frac{b}{a},$$
$$e_2 = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}, \quad e_3 = \alpha\beta\gamma = -\frac{d}{a}.$$

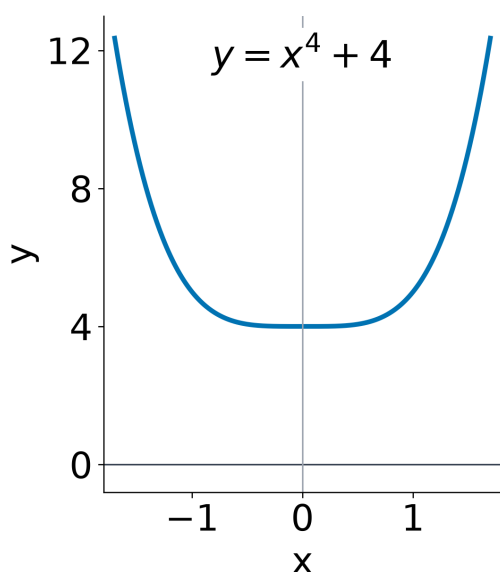
The signs alternate because each factor contains a minus sign. They do not say that the roots themselves are negative. For $(x - 1)^2(x - 3) = 0$, use the list 1, 1, 3, giving $e_1 = 5, e_2 = 7, e_3 = 3$. Counting only the two distinct values breaks the coefficient relations.

■ Include quartics and complex roots

For a **quartic equation** $ax^4 + bx^3 + cx^2 + dx + e = 0$, define e_1 as the sum of its four roots, e_2 as the sum of all six different pairs, e_3 as the sum of all four different triples, and e_4 as their product. Then

$$e_1 = -\frac{b}{a}, \quad e_2 = \frac{c}{a}, \quad e_3 = -\frac{d}{a}, \quad e_4 = \frac{e}{a}.$$

The six pairs are $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$: count each once. These are **symmetric functions**, unchanged when the roots exchange names.



No x-intercepts

Four complex roots

$$1 + i, \quad 1 - i$$

$$-1 + i, \quad -1 - i$$

Sum = 0

Product = 4

The four roots of $x^4 + 4 = 0$ are $1 + i, 1 - i, -1 + i, -1 - i$. Their sum is 0 and their product is 4, although the real graph has no x-intercepts.

■ Find sums of powers without solving the equation

Write $S_k = \alpha^k + \beta^k + \dots$, including each root with its multiplicity. Expanding the square and cube of the root sum gives

$$S_2 = e_1^2 - 2e_2, \quad S_3 = e_1^3 - 3e_1e_2 + 3e_3.$$

The first formula works for degrees 2, 3 and 4; the second is written here for degrees 3 and 4. For a higher sum, use the equation satisfied by each root. If $x^4 + bx^3 + cx^2 + dx + e = 0$, add its four root equations:

$$S_4 + bS_3 + cS_2 + dS_1 + 4e = 0.$$

Worked example. For $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$, the symmetric sums are 2, 3, 4, 5. Therefore $S_2 = 4 - 6 = -2$ and $S_3 = 8 - 18 + 12 = 2$. The original equation then gives $S_4 = 2(2) - 3(-2) + 4(2) - 4(5) = -2$. Negative sums of squares are possible when the roots are complex.

■ Recover unknown coefficients from given sums

For a cubic $x^3 + bx^2 + cx + d = 0$ with leading coefficient 1, first use $b = -S_1$ and $c = (S_1^2 - S_2)/2$. To connect the remaining coefficient to higher sums, add the three original root equations:

$$S_3 + bS_2 + cS_1 + 3d = 0.$$

Multiply each root equation by its own root and add again:

$$S_4 + bS_3 + cS_2 + dS_1 = 0.$$

The constant contributes $3d$ in the first equation but dS_1 in the second. Substitute the known sums, solve the resulting simultaneous equations and check in both relations.

■ Shift or scale all roots

To find an equation with roots $y = mx + k$, where $m \neq 0$, use the inverse **substitution** $x = (y - k)/m$. Substitute into the original polynomial, clear denominators and collect powers of y . A non-zero constant multiple of an equation has the same roots.

Worked example. Let the original polynomial be $P(x) = x^3 - 3x + 1$. For roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$, put $x = (y - 1)/2$ and multiply by 8:

$$(y - 1)^3 - 12(y - 1) + 8 = 0,$$

$$y^3 - 3y^2 - 9y + 19 = 0.$$

Its root sum is 3, also $2S_1 + 3(1) = 3$. For a degree- n polynomial, the corresponding sum check is $mS_1 + nk$. Do not substitute $2y + 1$ when the original variable should be $(y - 1)/2$.

■ Reciprocals and forbidden values

For new roots $y = 1/x$, put $x = 1/y$ and multiply by y^n . Check the original constant term is non-zero: a zero root has no reciprocal. A cubic $ax^3 + bx^2 + cx + d = 0$ becomes $dy^3 + cy^2 + by + a = 0$.

For example, $2x^3 - 3x^2 + 4x - 5 = 0$ gives reciprocal-root equation $5y^3 - 4y^2 + 3y - 2 = 0$. Its root sum is $4/5$, also e_2/e_3 . The new constant term excludes $y = 0$.

For $y = 1/(x - k)$, use $x = k + 1/y$ and check $P(k) \neq 0$ before claiming that every original root has an image. Clearing denominators does not define a reciprocal of zero.

■ Square the roots and preserve multiplicities

Separate a polynomial into its even and odd powers: $P(x) = E(x^2) + xO(x^2)$. If $y = x^2$, eliminate x by writing

$$E(y)^2 - yO(y)^2 = 0.$$

This combines the two possible signs of a square root: one of them solves the original equation. If different original roots have the same square, keep their repeated factors in the new equation.

Worked example. For $x^3 - 2x^2 - x + 2 = 0$, $E(y) = 2 - 2y$ and $O(y) = y - 1$. Thus $(2 - 2y)^2 - y(y - 1)^2 = 0$, which simplifies, up to a non-zero constant factor, to $(y - 1)^2(y - 4) = 0$. Original roots 1, -1, 2 become 1, 1, 4. Removing the repeated factor would lose one transformed root.

■ Simplify a higher-power transformation first

If $x^3 + 3x - 2 = 0$, each root satisfies $x^3 = 2 - 3x$. A new root $y = x^3 + 1$ is therefore $3 - 3x$. Use $x = (3 - y)/3$ in the original equation; multiplying by 27 and collecting terms gives $y^3 - 9y^2 + 54y - 54 = 0$. Now use coefficient relations on this new equation to find sums of transformed powers.

Rational functions and graphs

■ Check a fitted curve before trusting it

A program may fit a ratio of two **polynomials** to data. Its graph can look smooth on a screen while hiding an excluded input or a separate branch. For a **rational function** $f(x) = P(x)/Q(x)$, begin with the original **denominator**: require $Q(x) \neq 0$. Then find the features that a sketch must show.

■ An excluded input can be a hole

A denominator zero gives a **vertical asymptote** only when the remaining expression becomes unbounded there. Factor first; cancellation may instead leave a missing point. For $f(x) = (x^2 - 1)/(x - 1) = x + 1$, the original restriction $x \neq 1$ remains. Draw the line with an open point at (1, 2), not a vertical asymptote. A zero of the numerator is an intercept only if the original denominator is non-zero there.

■ Asymptotes from degree and division

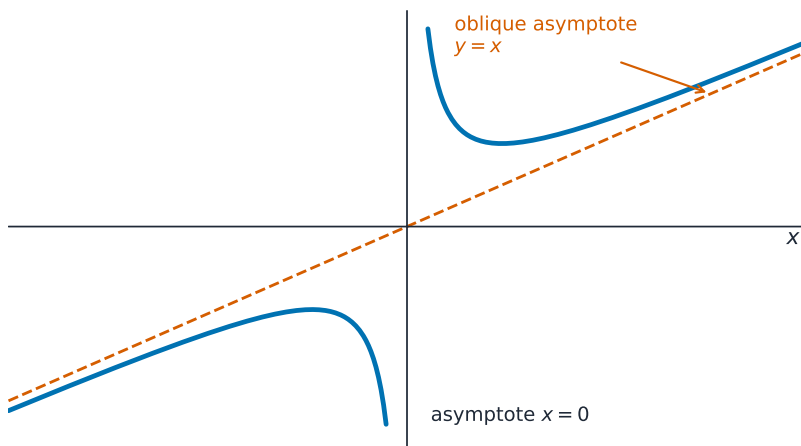
After cancellation, compare degrees: a lower-degree numerator gives horizontal limit 0; equal degrees give the ratio of leading coefficients. An **oblique asymptote** is a sloping line, so the numerator degree must be exactly one higher. Divide to obtain the **quotient** plus the **remainder** over the denominator. For $(x^2 - 1)/(x - 2)$, this is $x + 2 + 3/(x - 2)$. Thus $y = x + 2$ is the slant limit as $x \rightarrow \pm\infty$; $x = 2$ is vertical.

A horizontal or slant asymptote describes distant behaviour; it may be crossed at a finite input. For $f(x) = (x^2 + x)/(x^2 + 1)$, the horizontal asymptote is $y = 1$, and $f(1) = 1$.

■ Turning points and a complete sketch

For $f(x) = x + 1/x$, $x \neq 0$, $f'(x) = 1 - 1/x^2$. The **turning points** are (-1, -2), a local maximum, and (1, 2), a local minimum, as the derivative changes sign. There are no axis intersections. Draw both branches with vertical asymptote $x = 0$ and slant asymptote $y = x$. The left branch lies below the slant line and the right above it.

Near zero, the left branch falls to $-\infty$ and the right rises to $+\infty$.



oblique asymptote when $\deg(\text{num}) = \deg(\text{den}) + 1$ (long division)

For this curve the limiting lines, branch sides and two turning points determine the main shape; detailed plotting is unnecessary.

■ Check whether stationary points exist

For $h(x) = (x + 1)/(x^2 + 2x - 2)$, the quotient rule gives

$$h'(x) = -\frac{(x + 1)^2 + 3}{(x^2 + 2x - 2)^2}.$$

The numerator inside the minus sign is positive, and the denominator is positive at every allowed input. Therefore the derivative never vanishes. There are no stationary points, and each separate domain interval has a decreasing branch; do not join branches across a pole.

■ Prove the range with a discriminant

Rearrange $y = x + 1/x$ as $x^2 - yx + 1 = 0$. A real input exists exactly when its **discriminant** satisfies $y^2 - 4 \geq 0$. The range is $y \leq -2$ or $y \geq 2$, including both turning values. Check exclusions and any case where the equation stops being quadratic. For $y = x/(x^2 + 1)$, rearrangement gives $yx^2 - x + y = 0$. At $y = 0$ this is the valid linear equation $x = 0$; do not discard that range value.

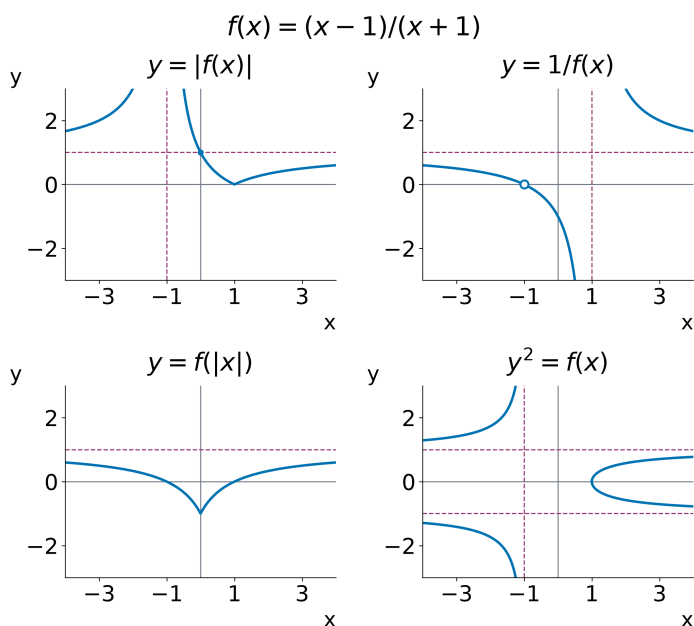
For the second function, $1 - 4y^2 \geq 0$ gives $-1/2 \leq y \leq 1/2$. Its denominator is never zero for real x , so all these values occur, including 0 and the two endpoints.

■ Change the output: modulus and reciprocal

Take $f(x) = (x - 1)/(x + 1)$. For $y = |f(x)|$, keep nonnegative pieces and reflect negative pieces above the x -axis. The exclusion $x = -1$ remains. Its only zero is $x = 1$. For $y = 1/f(x) = (x + 1)/(x - 1)$, exclude both $x = -1$ and $x = 1$: the first was never in the original domain and the second makes the reciprocal undefined. There is a hole at $(-1, 0)$ and a vertical asymptote at $x = 1$; the hole is not a zero.

■ Change the input, or square the output

For $f(|x|)$, keep the original right half and reflect it across the y -axis. Here the denominator $|x| + 1$ never vanishes; the range is $[-1, 1]$, with zeros at $x = \pm 1$ and a corner at $(0, -1)$. Such a corner is not automatic: if the original right-hand slope at zero is zero, the join can be smooth. For $y^2 = f(x)$, keep only original inputs where $f(x) \geq 0$, then draw both $y = \sqrt{f(x)}$ and $y = -\sqrt{f(x)}$. Here $x < -1$ or $x \geq 1$; the two branches meet at $(1, 0)$, and no graph exists for $-1 < x < 1$.



The open point belongs to neither reciprocal branch. Taking a modulus of the output and taking a modulus of the input give different domains and different zeros.

For a contrasting smooth join, take $h(x) = (x^2 + 4x + 8)/(x + 2)$. Since $h(0) = 4$ and $h'(0) = 0$, reflecting the right half gives matching slopes 0 on both sides: $h(|x|)$ is smooth at $(0, 4)$.

■ Solve inequalities from boundaries and branches

To solve $|(x - 1)/(x + 1)| < 2$, find where $f = 2$ and $f = -2$: the boundary inputs are -3 and $-1/3$. Include the pole -1 in the sign or branch diagram. Test the intervals separated by these values. The answer is $x < -3$ or $x > -1/3$, with strict endpoints excluded. Do not multiply an inequality by $x + 1$ without first deciding its

sign; multiplying by $(x + 1)^2$ is safe only after excluding $x = -1$.

■ Use symmetry to solve an input-modulus inequality

For $f(|x|) < 1/2$, put $t = |x| \geq 0$. Then $(t - 1)/(t + 1) < 1/2$ has a positive denominator and gives $t < 3$. Translate back to $-3 < x < 3$. The value zero is included. If a quadratic calculation produces a negative boundary for t , it cannot be a value of $|x|$; inspect the allowed half-line before reflecting the result.

■ Build the sketch from evidence

Record the domain, intercepts, asymptotes, stationary coordinates and one-sided behaviour. Use a range calculation to confirm which vertical gaps are real; a horizontal asymptote need not be an excluded output value. For a related graph, state the new domain before drawing. For an inequality, mark all boundaries and poles, test branches, and preserve strict or inclusive endpoints.

Summation of series

■ Count work without adding every stage

An algorithm performs r^2 operations at stage r . To estimate its total work after thousands of stages, we need one expression for the whole sum. **Summation** can combine standard formulas or cancel shifted terms. An infinite total needs a separate check: a finite formula alone does not prove convergence.

■ Standard sums and their bounds

For a positive integer n , the standard results are:

$$\sum_{r=1}^n r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{r=1}^n r^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

The limits matter: $\sum_{r=1}^n 1 = n$, but $\sum_{r=0}^n 1 = n + 1$. Check the number of terms before applying a formula. The cube identity uses consecutive integers from 1; it is not true for arbitrary lists of numbers.

■ Expand first, then sum each part

For $\sum_{r=1}^n (2r + 1)(r + 2)$, expand to $2r^2 + 5r + 2$. The result is $2 \sum r^2 + 5 \sum r + 2n$. Substitute the standard results and simplify:

$$\sum_{r=1}^n (2r + 1)(r + 2) = \frac{n(4n^2 + 21n + 29)}{6}.$$

At $n = 2$, the original terms are 3×3 and 5×4 , totalling 29. The formula also gives $2(16 + 42 + 29)/6 = 29$. A small- n check catches missing constants and index errors.

■ **Change the starting or ending index**

If $F(m) = \sum_{r=1}^m u_r$, then $\sum_{r=p}^q u_r = F(q) - F(p - 1)$. The last removed term is $p - 1$, not p . For $\sum_{r=0}^n (r + 1)^2$, put $k = r + 1$. The new bounds are 1 and $n + 1$, so the sum is $(n + 1)(n + 2)(2n + 3)/6$. Change the limits as well as the expression.

■ **The method of differences**

Write a term as $F(r) - F(r + 1)$, then show the first few terms and the last one. The middle terms **telescope**, leaving the boundary values $F(1) - F(n + 1)$. A wider shift leaves more boundary terms. For $F(r) - F(r + 2)$, the two starting terms and two ending terms survive:

$$\sum_{r=1}^n (F(r) - F(r + 2)) = F(1) + F(2) - F(n + 1) - F(n + 2).$$

Index	1	2	3	4	5	6	7	8
positive	$+\frac{1}{1}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{4}$	$+\frac{1}{5}$	$+\frac{1}{6}$		
negative			$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$	$-\frac{1}{6}$	$-\frac{1}{7}$	$-\frac{1}{8}$

$$S_6 = F(1) + F(2) - F(7) - F(8)$$

With $F(r) = 1/r$, the terms at indices 3 through 6 cancel in this six-term example. Both ending terms must remain.

■ **Partial fractions** **make cancellation visible**

For $1/[r(r + 2)]$, solve $1 = A(r + 2) + Br$: $A = 1/2, B = -1/2$. Thus each term is $\frac{1}{2}(1/r - 1/(r + 2))$. Use the two-step cancellation, keeping its factor:

$$S_n = \frac{3}{4} - \frac{1}{2(n + 1)} - \frac{1}{2(n + 2)}.$$

A decomposition is not yet a sum. Show enough first and last terms to establish the cancellation, especially when three partial fractions or an index shift are involved. Check the formula at $n = 1$: it gives $1/3$, the original first term.

■ Take the limit of the finite sum

A series is **convergent** when its partial sums approach a finite real limit. In the last example the two ending fractions tend to zero, so its **sum to infinity** is $3/4$. The remaining tail after n terms is $S_\infty - S_n$. Here it is $\frac{1}{2(n+1)} + \frac{1}{2(n+2)}$, which can be used to choose how many terms give a required accuracy.

Terms tending to zero is not enough. For $u_r = \sqrt{r+1} - \sqrt{r}$, the terms tend to zero, but $S_n = \sqrt{n+1} - 1$ grows without bound. A finite partial-sum limit is the deciding evidence.

■ Differences need not be partial fractions

Let $P_r = r(r+1)(r+2)$. Factor the common part to obtain $P_{r+1} - P_r = 3(r+1)(r+2)$. Hence a sum of $(r+1)(r+2)$ is $(P_{n+1} - P_1)/3$. Another route is to put two fractions over one denominator. For example, $1/r^2 - 1/(r+1)^2 = (2r+1)/[r^2(r+1)^2]$. Recognising the difference avoids trying to use a formula for $\sum 1/r^2$.

■ Check parameter endpoints separately

Let $u_r = t^{r(r+1)} - t^{(r+1)(r+2)}$. The finite sum is $S_n = t^2 - t^{(n+1)(n+2)}$. For $|t| < 1$, its limit is t^2 . At $t = 1$ or $t = -1$, every exponent is even and every term is zero: the infinite sum is 0, not 1. For $|t| > 1$, the ending power grows without bound. The convergence set is $[-1, 1]$, but the sum needs separate endpoint values.

■ Prove the sum before taking infinity

Choose standard sums after expanding, or differences after rearranging the term. State the index limits, show the surviving boundaries and check a small value of n . For an infinite series, inspect the limit of the exact partial sum. Check special parameter values in the original terms; use the remaining tail, not just the next term, for an error bound.

Matrices

■ Undo an image edit

A graphics program can store a rotation or stretch as a **matrix**. Its columns show where the two unit directions move; multiplying maps every point in the image. To undo the edit, an **inverse** must exist. To combine edits, their order matters. A correct answer should explain both the calculation and what happens to the shape.

■ Check the sizes before calculating

A matrix with m rows and n columns has size $m \times n$. Add or subtract only matrices of the same size, entry by entry; the **zero matrix** has every entry zero. If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$. Match the inner sizes, then multiply each row by each column. A product in the reverse order may have a different size or may

not exist.

For the non-square matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 2 \end{pmatrix},$$

$$AB = \begin{pmatrix} 1 & -4 \\ 9 & 1 \end{pmatrix}.$$

For example, entry $(1, 2)$ is $1(0) + 2(-1) + (-1)(2) = -4$. The product BA exists but is 3×3 , so it cannot equal AB .

■ Identity, order and singular matrices

The **identity matrix** I_n is $n \times n$, with ones on its main diagonal and zeros elsewhere. For A of size $m \times n$, $I_m A = A = A I_n$. Usually $AB \neq BA$; matrix multiplication is not commutative. A square matrix is **singular** when its **determinant**

is zero; it has no inverse. A non-singular square matrix has a unique inverse with $AA^{-1} = A^{-1}A = I$. A singular plane map may collapse the plane to a line or to a single point.

For $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$,

$$\det A = 3(4) - 1(2) = 10, \quad A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

For any 2×2 matrix, swap the two diagonal entries, negate the other two entries, then divide by $ad - bc$, provided this determinant is non-zero.

■ Expand a three by three determinant

To find a 3×3 determinant, choose a row and delete the row and column of each chosen entry. Each remaining 2×2 determinant is a **minor**. Across the first row the signs are $+, -, +$. For a parameter, factor the determinant and state every value making it zero. Do not divide by a parameter expression before checking whether it can vanish.

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Expanding across the first row,

$$\det M = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 0 = 1 - 2(-2) = 5.$$

The middle minus sign belongs to the position, even if the minor itself is negative.

■ Find and check a three by three inverse

A **cofactor** is a minor multiplied by its position sign. Use the sign pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$, then transpose the cofactor matrix: rows become columns.

Divide that transpose by the non-zero determinant. Check the result by multiplication. Computing one cofactor incorrectly or forgetting the transpose can give a plausible-looking matrix that is not an inverse.

For the matrix M above, the cofactor matrix and inverse are

$$C = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix},$$

$$M^{-1} = \frac{1}{5}C^T = \frac{1}{5} \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 4 & 1 \end{pmatrix}.$$

For example, the second row of M times the first column of M^{-1} gives $(0+2-2)/5 = 0$, as required for entry $(2, 1)$ of I_3 . Check all nine entries.

■ Undo several operations in reverse order

For non-singular square matrices of the same size, $(AB)^{-1} = B^{-1}A^{-1}$. Multiplication is associative, so $ABB^{-1}A^{-1} = I$. The reverse order removes the last operation first. Extend this to three factors: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. In a matrix equation, multiply on the correct side: from $AXB = C$, obtain $X = A^{-1}CB^{-1}$ when both inverses exist.

There is no general cancellation rule that moves a matrix through another matrix. Also, AB can be square and singular even when A and B are non-square: calculate the product before trying to take its determinant.

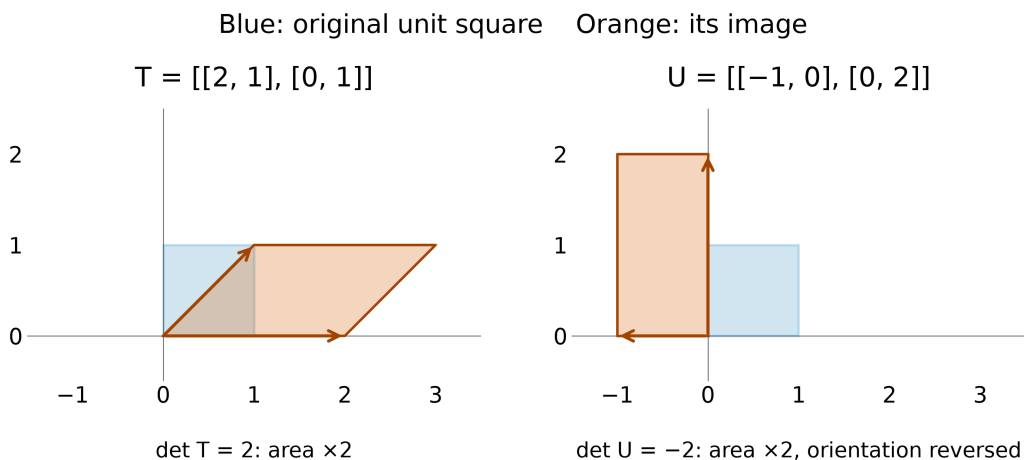
■ Build a plane transformation from columns

For column coordinates, $\begin{pmatrix} X \\ Y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}$. The first column is the image of $(1, 0)$ and the second is the image of $(0, 1)$. These determine a linear **transformation**; the origin stays fixed. Recognise rotation, reflection, **enlargement**, stretch and shear. State the centre or axis, direction and factor as appropriate. A translation with a non-zero displacement cannot be represented by a 2×2 matrix acting on these column coordinates.

Transformation	Matrix
Anticlockwise rotation through θ about the origin	$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$
Reflection in the x -axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Enlargement, centre origin, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Stretch parallel to the x -axis, factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Shear fixing the x -axis, $(0, 1) \mapsto (h, 1)$	$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$

■ Compose edits and interpret area

The product AB applies B first, then A . A stretch $S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ followed by a 90° anticlockwise rotation R gives $RS = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$. Reversing them gives $SR = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$. The area scale factor is $|\det T|$. The sign of $\det T$ tells whether orientation is reversed, not whether an area is negative. A negative enlargement factor reverses both coordinate directions, so its determinant k^2 is positive.



Both images have area 2. Their determinants have opposite signs: area magnitude and orientation are different properties.

■ Find points that stay fixed

An **invariant point** keeps its own coordinates: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Solve $(T - I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The origin is always a solution; a line of fixed points requires more solutions than the origin alone. For $T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, the equations reduce to $x + y = 0$. Every point on $y = -x$ stays fixed. If $\det(T - I) \neq 0$, the origin is the only invariant point; if it is zero, solve the equations to find the actual set.

For $T_\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} R_\theta$, a non-zero fixed point requires

$$\det(T_\theta - I) = 2 - 2\cos\theta - \sin\theta = 0.$$

On $0 < \theta < 2\pi$, put $t = \tan(\theta/2)$, checking $\theta = \pi$ separately. Then $4t^2 - 2t = 0$, giving $\theta = 2\arctan(1/2)$. Substitution gives $T_\theta - I = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, so the fixed line is $y = 2x$. The determinant condition alone does not give its equation.

■ A line can stay invariant while its points move

An **invariant line** maps onto itself as a set. Its individual points may move along it. For the non-singular map $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, put $y = mx$ and require $Y = mX$ for every point on that line. This gives $bm^2 + (a - d)m - c = 0$. State equations of lines, not just slopes, and test the vertical line $x = 0$ separately: assuming a finite gradient can miss it.

For $T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ the finite-gradient equation is $-m - 1 = 0$, so $y = -x$ is invariant. The vertical line $x = 0$ is also invariant, because $(0, y) \mapsto (0, 3y)$.

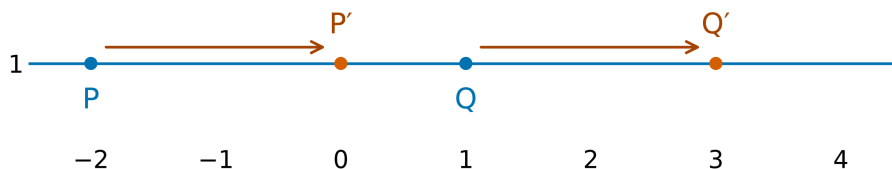
No non-zero point on $x = 0$ stays fixed. For this matrix, solving $(T - I)(x, y)^T = 0$ gives only the origin: invariant lines and lines of invariant points answer different questions.

■ Check lines away from the origin

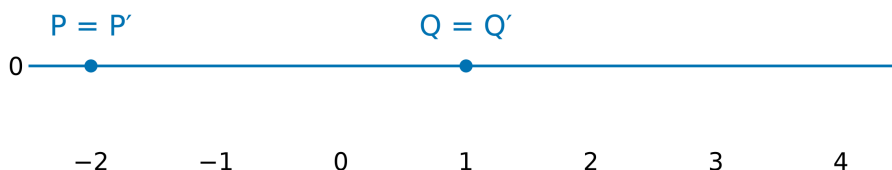
For a line $y = mx + c$, substitute its general point into (X, Y) and require $Y = mX + c$ for all x . Compare the coefficient of x and the constant term. Test vertical lines $x = c$ separately. For the shear $H = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, every horizontal line $y = c$ is invariant: $(x, c) \mapsto (x + 2c, c)$. Only $y = 0$ is a line of invariant points. On $y = 1$, every point moves two units right.

$$H = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}: (x, y) \rightarrow (x + 2y, y)$$

Invariant line: points move along it



Line of invariant points: each point stays fixed



The two marked points move along $y=1$, so the line stays invariant. Every point on $y=0$ stays in its own position.

■ Choose the condition before the algebra

First identify whether the task asks for a product, an inverse, a transformation, an invariant point or an invariant line. State sizes and restrictions before calculating, and describe geometric transformations fully. Verify an inverse by a product, a fixed point by $T\mathbf{v} = \mathbf{v}$, and an invariant line by its equation after transformation. A numerical check of one point cannot prove that an entire line is invariant.

The identity matrix is also called the **unit matrix**. An **inverse matrix** undoes a non-singular map; a **singular matrix** has no inverse. When describing a **geometric transformation**, distinguish an invariant point from an invariant line.

Polar coordinates

Polar curves are described by a radius and angle, together with an allowed interval.

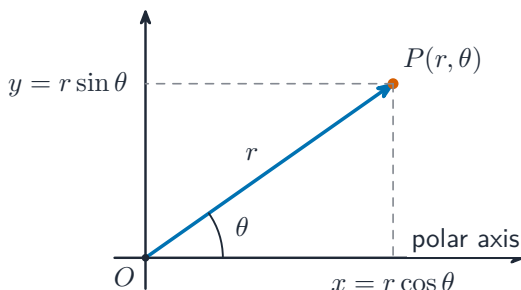
■ Choose coordinates that fit the measurement

A radar measures distance and direction. **Polar coordinates** store these as (r, θ) , rather than the horizontal and vertical distances used by **Cartesian coordinates**. A circle centred at the radar has the simple equation $r = a$. Other circles need different equations: the choice of origin still matters. Even with **rotational symmetry**, the permitted interval may show only part of a curve; read the interval too.

■ Locate a point and choose its angle

The **pole** is the origin; the **initial line** points along the positive x -axis. Use $r \geq 0$ in this course. Angles are measured anticlockwise from the initial line; use radians for calculus. Convert with $x = r \cos \theta$, $y = r \sin \theta$ and $r^2 = x^2 + y^2$. In the reverse direction, use $r = \sqrt{x^2 + y^2}$ and select an angle in the stated interval using the quadrant, not just $\arctan(y/x)$.

For $(-\sqrt{3}, 1)$, $r = 2$. An angle in $0 \leq \theta < 2\pi$ is $5\pi/6$, not $-\pi/6$: the point lies in quadrant II. At the pole $r = 0$, all angle descriptions name the same point, so the angle is not unique.



The radius is a distance. Its horizontal and vertical components may be negative even though r is non-negative.

■ Convert equations and keep their domains

Substitute the coordinate relations into an equation, simplify, then check any step that multiplied or divided by r . Dividing by r can lose the pole; multiplying by r can introduce it. For $r = 4 \cos \theta$, multiplying by r gives $x^2 + y^2 = 4x$, so $(x - 2)^2 + y^2 = 4$. The circle has centre $(2, 0)$ and radius 2, traced once on $-\pi/2 \leq \theta \leq \pi/2$. A negative value from $4 \cos \theta$ is not permitted under $r \geq 0$.

The line $x = 2$ becomes $r \cos \theta = 2$, or $r = 2 \sec \theta$ with $-\pi/2 < \theta < \pi/2$. Multiplying by r would give $r^2 \cos \theta = 2r$, which also admits $r = 0$: that extra point is not on $x = 2$.

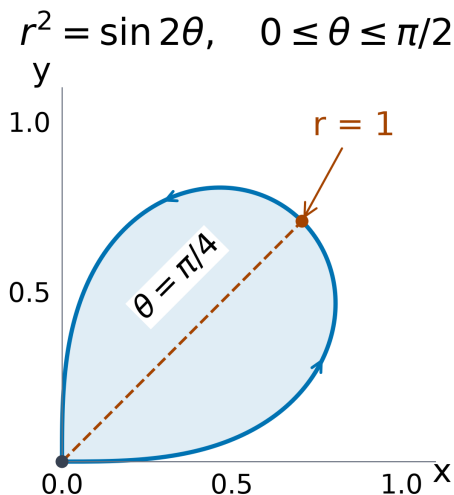
■ Build a sketch from features

Mark the pole and initial line, the stated angle interval, points on the initial line, zero and greatest/least radii, and the shape near the pole. Use a few special angles to place the curve; detailed point-by-point plotting is unnecessary. Test symmetry using both the equation and the interval. Replacing θ by $-\theta$ tests reflection in the initial line; reflection in a line at angle α replaces θ by $2\alpha - \theta$. A symmetric formula on an asymmetric interval need not give a symmetric whole sketch.

For $r = 2(1 + \cos \theta)$ on $-\pi \leq \theta \leq \pi$, the curve is a **cardioid**. It is symmetric about the initial line, reaches greatest radius 4 at $\theta = 0$, crosses the y -axis at $(0, \pm 2)$ and has a cusp at the pole when $\theta = \pm\pi$. It is one closed curve, not a circle.

■ A restricted interval can select one loop

If r^2 is given, take the non-negative square root and check where it is real. For $r^2 = \sin 2\theta$ on $0 \leq \theta \leq \pi/2$, the curve starts and ends at the pole, stays in the first quadrant and reaches $r = 1$ at $\theta = \pi/4$. Since $\sin(2(\pi/2 - \theta)) = \sin 2\theta$, this loop is symmetric about $\theta = \pi/4$. Near $\theta = 0$, $r^2 \approx 2\theta$, so it leaves the pole along the initial line; near the other endpoint it returns along the positive y -direction.



Both endpoints meet at the pole (0, 0).

Both endpoints are the same pole. The greatest radius lies on the symmetry line, not on either coordinate axis.

■ Find an area from radial sectors

A thin **sector** has area approximately $\frac{1}{2}r^2 \Delta\theta$. Adding sectors gives the exact area below, when the selected region is swept once. Square r , retain the factor $1/2$ and use radian limits. For the loop above, r^2 is already $\sin 2\theta$: do not square it again. The endpoints give the bounds, not a memorised full turn.

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$

$$A = \frac{1}{2} \int_0^{\pi/2} \sin 2\theta d\theta = \left[-\frac{1}{4} \cos 2\theta \right]_0^{\pi/2} = \frac{1}{2}.$$

For a complete cardioid $r = a(1 + \cos \theta)$, use $(1 + \cos \theta)^2$ and $\cos^2 \theta = (1 + \cos 2\theta)/2$. Its total area is $3\pi a^2/2$.

■ Bound a region between two curves

Find their common rays by solving their radii equal, and check the pole separately. Identify which radius is outer throughout each interval. The radial strip has area $\frac{1}{2}(r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta$; it is not the square of the difference of radii. In the first quadrant, the region inside $r = 4 \cos \theta$ but outside $r = 2$ lies on $0 \leq \theta \leq \pi/3$. The whole region also has a reflected lower half; include it only if the question asks for it.

$$A_{\text{upper}} = \frac{1}{2} \int_0^{\pi/3} (16 \cos^2 \theta - 4) d\theta = [2\theta + 2 \sin 2\theta]_0^{\pi/3} = \frac{2\pi}{3} + \sqrt{3}.$$

The pole is not a common point: $r = 2$ never reaches it. A point on one curve is not an intersection unless it lies on the other.

■ Furthest from the pole is not highest

Distance from the pole is r . To find its extrema, compare the allowed endpoints and stationary values of r , or of r^2 when $r \geq 0$. A point with largest r need not have largest x or y . Distance from the initial line for a first-quadrant point is $y = r \sin \theta$. Write the required distance before differentiating, then check its endpoints. For other regions, account for signs and whether the question names a full line or a half-line.

For $r = 2 \sin 2\theta$, $0 \leq \theta \leq \pi/2$, the largest radius is 2 at $\theta = \pi/4$. However,

$$y = 4 \sin^2 \theta \cos \theta.$$

Writing $u = \cos \theta$ gives $y = 4u(1 - u^2)$ for $0 \leq u \leq 1$. Its maximum occurs at $u = 1/\sqrt{3}$, giving $y_{\text{max}} = 8/(3\sqrt{3})$ at a different angle. The endpoints both give $y = 0$.

■ Integrate an unfamiliar polar curve

A polar equation need not be a named circle or cardioid. For $r = \theta e^{\theta/2}$ on $0 \leq \theta \leq \pi$, the radius increases from 0 to $\pi e^{\pi/2}$. The curve starts at the pole, turns anticlockwise through half a turn and ends on the negative x -axis. The region swept from the pole is bounded by the curve and the endpoint rays. Its area is $\frac{1}{2} \int_0^\pi \theta^2 e^\theta d\theta$. Integration by parts twice gives an exact value; do not replace it with the area of a circle having the largest radius.

$$\int \theta^2 e^\theta d\theta = e^\theta (\theta^2 - 2\theta + 2) + C,$$

$$A = \frac{1}{2} e^\pi (\pi^2 - 2\pi + 2) - 1.$$

■ Read the region before choosing limits

State the non-negative-radius convention and allowed angles, sketch the important features, and keep any lost or introduced pole under review when converting equations. For area, identify the region and sweep it once. For a greatest distance, decide whether the expression to optimise is r , x , y or a distance to the specified line. Check endpoints as well as stationary values.

Vectors

Vectors describe both magnitude and direction.

■ Find the shortest safe route in space

A drone must approach a flat roof by the shortest route. A cable must join two rails that do not meet. These are different geometry problems: first identify the points, lines and **plane** involved. A direction alone is not a complete line. A distance alone does not locate its endpoints. **Vector products** and scalar products let you construct the geometry and then check it.

■ Choose the product that answers the question

The scalar product is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. For non-zero vectors it equals $|\mathbf{u}||\mathbf{v}| \cos \theta$, so a zero scalar product identifies **perpendicular** directions. The zero vector has no direction or defined angle. The vector product is **anti-commutative**: swapping the order reverses its sign. Its magnitude is $|\mathbf{u}||\mathbf{v}| \sin \theta$, the area of the **parallelogram** spanned by the vectors. A triangle has half this area. For non-parallel inputs, $\hat{\mathbf{n}}$ is the unit normal in the ordered product direction.

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin \theta \hat{\mathbf{n}}, \quad |\hat{\mathbf{n}}| = 1,$$

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

For $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (0, 1, 1)$,

$$\mathbf{u} \times \mathbf{v} = (1, -1, 1), \quad \mathbf{u} \cdot \mathbf{v} = 1.$$

The cross product has zero scalar product with each input and magnitude $\sqrt{3}$. Parallel vectors have zero cross product, so they cannot supply a normal direction to a unique plane. For comparison, $(1, 1, 0)$ and $(1, -1, 0)$ have zero scalar product and are perpendicular.

■ Read the three forms of a plane

A **normal vector** is a non-zero vector perpendicular to every direction in the plane. The Cartesian form $ax + by + cz = d$ is the scalar-product form $\mathbf{r} \cdot \mathbf{n} = d$, with $\mathbf{n} = (a, b, c)$. A unit normal is $\mathbf{n}/|\mathbf{n}|$. The number d is not generally the distance from the origin unless the normal is a unit vector. In $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\mathbf{a}$ locates a point and the non-parallel vectors $\mathbf{b}, \mathbf{c}$ give directions in the plane. The parameters vary independently. Parallel direction vectors describe only a line, not a plane.

A plane with normal $(2, -1, 3)$ through $(1, 0, 2)$ has

$$2x - y + 3z = 2(1) - 0 + 3(2) = 8.$$

Multiplying every coefficient, including the right side, by the same non-zero constant gives the same plane.

■ Convert a parametric plane to Cartesian form

Take the cross product of its two direction vectors to find a normal. Then use a known point to find the constant: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$. For three given points, first form two displacement vectors from one of them. To return to a parametric equation, choose two coordinates as free parameters, solve for the third and separate a fixed point from the two direction terms. Check both directions have zero scalar product with the normal.

For

$$\mathbf{r} = (1, 0, 2) + \lambda(1, 1, 0) + \mu(0, 1, 1),$$

the normal is $(1, -1, 1)$, giving $x - y + z = 3$. Conversely, set $x = 1 + \lambda$ and $z = 2 + \mu$. The Cartesian equation gives $y = \lambda + \mu$, recovering the original parametric form.

■ Classify a line and a plane before solving

Substitute $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ into $\mathbf{n} \cdot \mathbf{r} = d$. If $\mathbf{n} \cdot \mathbf{u} \neq 0$, there is one intersection, found from the parameter below. Substitute back to give the actual point. If $\mathbf{n} \cdot \mathbf{u} = 0$, test the base point. When $\mathbf{n} \cdot \mathbf{a} = d$, the whole line lies in the plane; otherwise it is parallel and separate. Dividing by zero would hide this distinction.

$$t = \frac{d - \mathbf{n} \cdot \mathbf{a}}{\mathbf{n} \cdot \mathbf{u}} \quad \text{when } \mathbf{n} \cdot \mathbf{u} \neq 0.$$

For the plane $x - y + z = 3$, the line $\mathbf{r} = t(1, 0, 1)$ meets it when $2t = 3$, at $(3/2, 0, 3/2)$. The line $(1, 0, 2) + t(1, 1, 0)$ lies in the plane. The line $t(1, 1, 0)$ is parallel and separate, because its points satisfy $x - y + z = 0$ instead.

■ Find the foot of a perpendicular

The foot H from a point P to a plane lies on the normal line through P . Write $\mathbf{h} = \mathbf{p} - t\mathbf{n}$, substitute into the plane equation, then solve for t . The sign determines which way to travel along the normal. Verify both conditions: H lies in the plane and $\overrightarrow{PH}$ is parallel to its normal. Merely finding some point in the plane does not make it the nearest point.

From $P = (3, 0, 3)$ to $x - y + z = 3$, use $\mathbf{n} = (1, -1, 1)$:

$$\mathbf{n} \cdot (\mathbf{p} - t\mathbf{n}) = 3 \quad \Rightarrow \quad 6 - 3t = 3.$$

Thus $t = 1$ and $H = (2, 1, 2)$. It satisfies $2 - 1 + 2 = 3$, and $\overrightarrow{PH} = (-1, 1, -1)$ is parallel to the normal.

■ Measure a point-to-plane distance

The shortest distance is the component of the displacement in the normal direction. For a non-zero normal, use the absolute numerator to obtain a non-negative distance; do not confuse the plane constant with the distance symbol. Multiplying a plane equation by a non-zero constant changes numerator and denominator by the same magnitude, leaving the distance unchanged. A zero numerator means the point already lies in the plane.

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|\mathbf{n} \cdot \mathbf{p} - d|}{|\mathbf{n}|}.$$

For $P = (3, 0, 3)$ and $x - y + z = 3$, $D = |6 - 3|/\sqrt{3} = \sqrt{3}$. This agrees with the length of $\overrightarrow{PH}$ above. The plane's distance from the origin is also $|3|/\sqrt{3} = \sqrt{3}$, although these are different point-to-plane problems.

■ Use normals for plane angles

The smaller angle between two planes is the smaller angle between their normal directions, with $0 \leq \alpha \leq \pi/2$. The absolute scalar product removes the choice of normal direction: $\mathbf{n}$ and $-\mathbf{n}$ describe the same plane. The angle ϕ between a line and a plane is complementary to the angle between its direction and the plane normal. Use sine for the line-plane angle, with $0 \leq \phi \leq \pi/2$, and check both direction vectors are non-zero.

$$\cos \alpha = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}, \quad \sin \phi = \frac{|\mathbf{u} \cdot \mathbf{n}|}{|\mathbf{u}||\mathbf{n}|}.$$

The planes $x - y + z = 3$ and $x + y + z = 5$ have normals with scalar product 1 and lengths $\sqrt{3}$, so $\alpha = \arccos(1/3) \approx 70.5^\circ$. A line with direction $(1, 0, 1)$ makes angle $\phi = \arcsin(2/\sqrt{6}) \approx 54.7^\circ$ with the first plane. Using cosine would give its angle with the normal instead.

■ Find the intersection of two planes

For non-parallel normals, the intersection direction is $\mathbf{n}_1 \times \mathbf{n}_2$. Find a common point by solving both plane equations, then give a full line equation: point plus parameter times direction. If the normals are parallel, compare the constants using the same scale factor: the planes are either identical or parallel and separate. A zero cross product cannot provide an intersection direction.

Subtract $x - y + z = 3$ from $x + y + z = 5$ to get $y = 1$, then $x + z = 4$. A common point is $(4, 1, 0)$. The cross product of the normals is $(-2, 0, 2)$, so an equivalent intersection line is

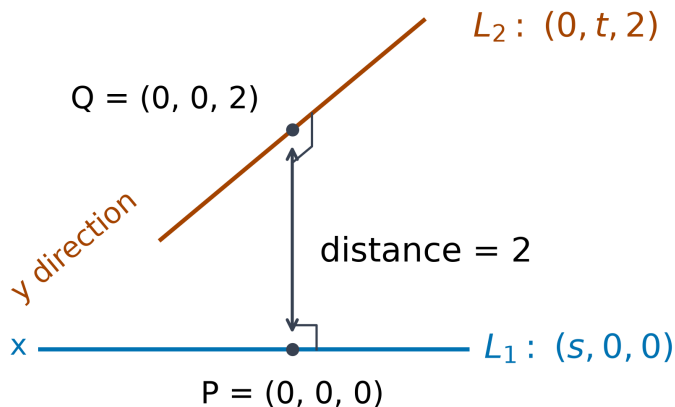
$$\mathbf{r} = (4, 1, 0) + t(-1, 0, 1).$$

Every point of this line satisfies both plane equations; there is no need to use the unsimplified direction vector.

■ Decide whether two lines are skew

Skew lines are neither parallel nor intersecting. For $\mathbf{r} = \mathbf{a} + s\mathbf{u}$ and $\mathbf{r} = \mathbf{b} + t\mathbf{v}$, compare their directions and, if non-parallel, solve all three coordinate equations for an intersection. Two different line parameters must be used. When $\mathbf{u} \times \mathbf{v} \neq 0$, the lines are coplanar exactly when $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v}) = 0$. A non-zero value proves they are skew. Do not apply the skew-distance denominator to parallel lines.

Two skew lines: different heights and different directions



PQ is perpendicular to both lines in three-dimensional space.

This schematic uses the x-axis and a y-parallel line at height 2. Their nearest points have the same x and y coordinates, so their connector has length 2.

■ Calculate the shortest skew-line distance

For non-parallel lines, $\mathbf{u} \times \mathbf{v}$ is perpendicular to both. Project the displacement between their base points onto this direction. Any non-zero multiple of the cross product gives the same distance. The formula below gives a distance, not the location of the shortest segment. For parallel lines use $|(\mathbf{b} - \mathbf{a}) \times \mathbf{u}|/|\mathbf{u}|$ instead; for intersecting non-parallel lines the numerator is zero.

$$D = \frac{|(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.$$

Let

$$L_1 : \mathbf{r} = (1, 0, 1) + s(1, 1, 0), \quad L_2 : \mathbf{r} = (2, 2, 3) + t(0, 1, 1).$$

Their cross product is $(1, -1, 1)$ and their base-point displacement is $(1, 2, 2)$. Its scalar product with the cross product is 1, so they are skew and $D = 1/\sqrt{3}$.

■ Locate the common perpendicular

Let $P = \mathbf{a} + s\mathbf{u}$ and $Q = \mathbf{b} + t\mathbf{v}$. The shortest connector obeys $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{u} = 0$ and $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} = 0$. Solve these two simultaneous equations for the parameters. Give the common perpendicular as a line through P in direction $\mathbf{u} \times \mathbf{v}$, or through Q in the same direction. Check it meets both original lines and that the segment length agrees with the distance formula. A perpendicular direction through an arbitrary point is insufficient.

For the two lines above,

$$\mathbf{q} - \mathbf{p} = (1 - s, 2 + t - s, 2 + t).$$

Orthogonality gives $2s - t = 3$ and $s - 2t = 4$, hence $s = 2/3$ and $t = -5/3$. Therefore

$$P = (5/3, 2/3, 1), \quad Q = (2, 1/3, 4/3),$$

$$\mathbf{r} = (5/3, 2/3, 1) + k(1, -1, 1)$$

is the common perpendicular. At $k = 1/3$ it reaches Q , and $|PQ| = \sqrt{3}/3$ agrees with $1/\sqrt{3}$.

■ Check the geometric conditions

A normal must be non-zero, a plane needs two independent directions, and the skew-distance denominator requires non-parallel lines. State parameter restrictions before squaring distance equations; absolute values can give more than one valid value. Substitute a proposed point into its plane, check an intersection line in both planes, and verify both orthogonality equations for a common perpendicular. These checks explain why the result solves the geometry.

Proof by induction

A proof using this method is an **inductive proof**.

■ A repeatable rule must start somewhere

Checking a pattern for the first few **positive integers** cannot establish infinitely many cases. **Mathematical induction** joins a first true statement to a rule that reaches every next statement. The **base case** starts the chain. The **inductive step** proves a conditional link for an arbitrary allowed integer, not just the next number in a short table.

■ State exactly what is being proved

Write the claim $P(n)$ and its integer range, starting at n_0 . Check $P(n_0)$ directly, using a given starting condition if appropriate. Then assume $P(k)$ for an arbitrary integer $k \geq n_0$ and use that assumption to derive $P(k+1)$. The **inductive hypothesis** is temporary and conditional: it does not assume the result for every integer. Conclude that $P(n)$ holds for all integers $n \geq n_0$, because the verified base case and the general link together reach them.

Do not assume the $k+1$ case while trying to prove it. Showing only $P(k+1) \Rightarrow P(k)$ gives the wrong implication. Backward algebra can help you plan, but a proof needs a valid forward argument, or a justified chain of equivalent statements.

■ Add the new term in a sum

For a finite sum, separate the last term of the $k+1$ sum. Replace only its first k terms using the hypothesis, then simplify to the formula with $n = k+1$. Show the starting case, the exact hypothesis and the final conclusion. A line that merely quotes the sum formula for $k+1$ has not established the link.

Prove $\sum_{r=1}^n r(r+1) = n(n+1)(n+2)/3$ for positive integers n .

At $n = 1$, both sides equal 2. Assume the formula holds for $n = k$. Then

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+1) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

This is the required $k+1$ formula, so the result follows for all positive integers by induction.

■ Expose an integer multiple for divisibility

To prove an expression is **divisible** by a non-zero integer m , write it as m times an integer. In the inductive step, isolate the assumed divisible expression and show every remaining term is also a multiple of m . Both a convenient multiple of the k expression and the difference between consecutive expressions can help. Name

the integer multiplier; “it is divisible by inspection” can hide a missing use of the hypothesis.

Let $F_n = 10^n - 4^n$. At $n = 1$, $F_1 = 6$. Assume $F_k = 6q$ for an integer q . Then

$$\begin{aligned} F_{k+1} &= 10(10^k - 4^k) + 6 \cdot 4^k \\ &= 6(10q + 4^k). \end{aligned}$$

The bracket is an integer. Thus 6 divides F_{k+1} , and induction proves the result for every positive integer n .

■ Use the recurrence to move forward

A recurrence gives a legal way to pass from u_k to u_{k+1} . Check that a proposed closed form has the correct first term, substitute its k version into the recurrence and simplify. Do not insert the proposed $k + 1$ formula into the recurrence and call it a proof unless you establish a reversible equivalence with the known k formula. The clearest route starts from the given recurrence.

Suppose $u_1 = 2$ and $u_{n+1} = 3u_n + 2$. The claim is $u_n = 3^n - 1$. It is true at $n = 1$. Assuming $u_k = 3^k - 1$,

$$u_{k+1} = 3(3^k - 1) + 2 = 3^{k+1} - 1.$$

This proves the claimed closed form for all positive integers, not just the first computed terms 2, 8, 26.

■ Preserve a bound and check denominator signs

For an inequality involving a recurrence, substitute the hypothesis into an expression whose sign can be controlled. State why its denominator is positive before using it to preserve an inequality. Sometimes the needed hypothesis contains two bounds. Prove both survive the step. Once a bound is established for every term, it can justify a later monotonicity argument without assuming the answer.

Let $u_1 = 1$ and $u_{n+1} = (4u_n + 3)/(u_n + 2)$. Prove $0 < u_n < 3$.

The first term satisfies both inequalities. If $0 < u_k < 3$, numerator and denominator of u_{k+1} are positive. Also,

$$3 - u_{k+1} = \frac{3 - u_k}{u_k + 2} > 0.$$

Thus $0 < u_{k+1} < 3$. By induction, $0 < u_n < 3$ for every positive integer n . The established bounds then give

$$u_{n+1} - u_n = \frac{(3 - u_n)(u_n + 1)}{u_n + 2} > 0,$$

so the sequence is increasing. This last deduction uses the proven bound; it does not replace that proof.

■ Prove a matrix-power formula by multiplication

For powers of one matrix, $M^{k+1} = M^k M$. Substitute the proposed expression for M^k , multiply row by column and show every entry has the $k + 1$ form. A matrix power is not found by raising each entry to a power. Also, a formula checked at M and M^2 is only evidence until the arbitrary- k step has been proved.

For $M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, claim

$$M^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix} \quad (n \geq 1).$$

The $n = 1$ case is M itself. Using the k hypothesis,

$$\begin{aligned} M^{k+1} &= \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore the formula holds for all positive integers by induction. The separate $n = 0$ case is $M^0 = I$, which also fits this formula.

■ Conjecture and prove a derivative pattern

A **conjecture** is a statement suggested by evidence, not an established theorem. Compute a few derivatives, look for a pattern and express it precisely with its starting value of n . For an n th-derivative formula, differentiate the assumed k th derivative once. A correct pattern needs an actual differentiation and simplification in the inductive step. For $f(x) = xe^{2x}$, the first derivatives are $e^{2x}(2x + 1)$, $e^{2x}(4x + 4)$ and $e^{2x}(8x + 12)$. Conjecture

$$f^{(n)}(x) = 2^{n-1}e^{2x}(2x + n), \quad n \geq 1.$$

The first derivative gives the base case. Differentiating the k formula yields

$$2^{k-1}e^{2x}\{2(2x + k) + 2\} = 2^k e^{2x}(2x + k + 1),$$

which is the required $k + 1$ derivative. Hence the conjecture is true for every positive integer n .

■ Match the derivative order in the statement

If the claimed formula is for derivative order $2n$, increasing n by one increases the derivative order by two. The inductive step must differentiate twice. Count the derivative order before starting the algebra. Track the alternating sign and the coefficient depending on n separately. A statement about selected even orders is not automatically a statement about every derivative order.

For $f(x) = x \sin x$, claim

$$f^{(2n)}(x) = (-1)^n(x \sin x - 2n \cos x), \quad n \geq 1.$$

At $n = 1$, $f''(x) = 2 \cos x - x \sin x$, as required. Differentiating the assumed $2k$ formula once gives $(-1)^k\{x \cos x + (2k + 1) \sin x\}$. A second differentiation gives

$$(-1)^k\{-x \sin x + (2k + 2) \cos x\} = (-1)^{k+1}\{x \sin x - 2(k + 1) \cos x\},$$

completing the step to derivative order $2(k + 1)$. The formula therefore holds for all positive integers n by induction.

■ Turn a short trial into a factorial conjecture

A **factorial** is $n! = 1 \cdot 2 \cdots n$ for a positive integer n , with $0! = 1$. A few partial sums may suggest a shifted factorial, but the pattern must then survive a general induction step. Compute the first terms carefully and look for a simple expression after adding or subtracting a constant. When a question asks for induction, present that proof even if a direct telescoping argument is also available.

For $S_n = \sum_{r=1}^n r r!$, the first sums are 1, 5, 23, 119. Adding 1 suggests $S_n = (n + 1)! - 1$. At $n = 1$, both sides equal 1. Assuming the formula for k ,

$$\begin{aligned} S_{k+1} &= (k + 1)! - 1 + (k + 1)(k + 1)! \\ &= (k + 2)(k + 1)! - 1 = (k + 2)! - 1. \end{aligned}$$

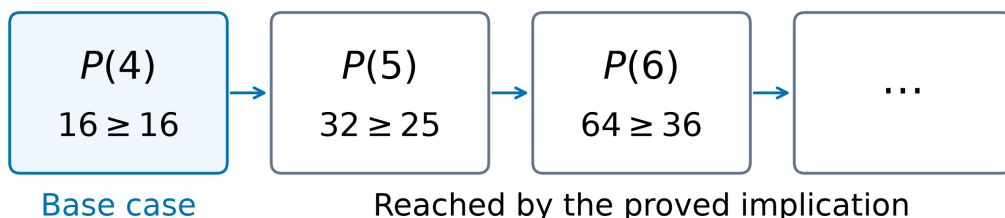
Thus the conjecture holds for all positive integers by induction. The identity $r r! = (r + 1)! - r!$ also explains why the sums telescope, but listing early values alone is not a proof.

■ Start at the first value the claim actually covers

A claim may start at $n = 4$ or another stated integer. Prove that starting case and assume k lies in the allowed range; do not silently replace the domain by all positive integers. For $2^n \geq n^2$ with $n \geq 4$, the base case is $16 = 16$. Assuming $2^k \geq k^2$ for $k \geq 4$, we have $2^{k+1} \geq 2k^2 \geq (k + 1)^2$, because $2k^2 - (k + 1)^2 = (k - 1)^2 - 2 > 0$ in that range. Hence induction proves the stated claim for all integers $n \geq 4$.

Claim: $2^n \geq n^2$ for integers $n \geq 4$

For every $k \geq 4$: $P(k) \Rightarrow P(k+1)$



Checked values alone do not justify the arrows.

The chain proves the stated range beginning at 4. In fact $n=3$ fails, since 8 is less than 9; the starting point is part of the claim.

■ Audit the implication and the conclusion

Check that the hypothesis is used to prove the next case in the forward direction, with all expressions defined. A checked base and a backward implication alone do not establish the claim for later integers. Match the conclusion to the link: $P(k) \Rightarrow P(k+1)$ reaches every later integer, while a link from k to $k+2$ reaches only one parity from a single base. To cover both parities with that link, two starting cases are needed.

The statements $P(n) : n \leq 5$ give a warning. $P(1)$ is true and $P(k+1) \Rightarrow P(k)$ is true for positive integers, yet $P(6)$ is false. The reversed implication cannot replace the required forward step.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k + 1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.