

3.5 Linear motion under a variable force

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS differential equation • variable force

Objective

Build the skills to answer exam questions on **linear motion under a variable force**

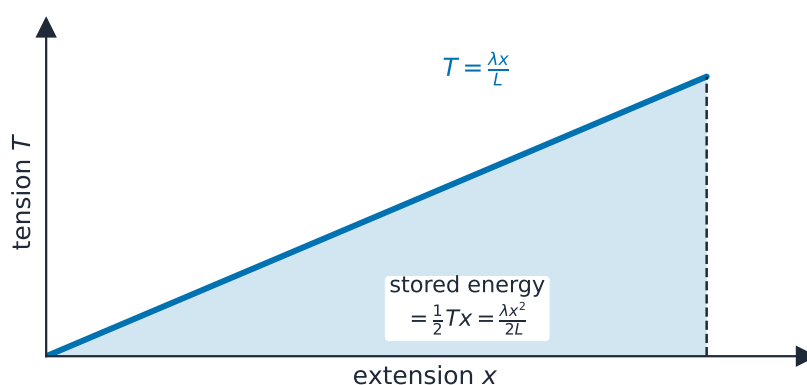
– using $a = v \frac{dv}{dx}$ (force depends on position) or $a = \frac{dv}{dt}$ (force depends on time). Take $g = 10 \text{ m s}^{-2}$.

You must be able to:

- choose $a = v \frac{dv}{dx}$ for a position-dependent force
- set up and solve the resulting differential equation
- use $a = \frac{dv}{dt}$ for a time-dependent force

1 Key knowledge and worked examples

■ Force depending on position



$$T = \frac{\lambda}{L}x \quad \cdot \quad E = \frac{1}{2}Tx = \frac{\lambda x^2}{2L} \quad (\text{area under graph})$$

A position-dependent force (e.g. a spring) makes Newton's law a differential equation in v and x

8 kg under force $(x^3 + 4x)$ N, $v = 1$ at $x = 0$: $8v \frac{dv}{dx} = x^3 + 4x$; integrate $\rightarrow 4v^2 = \frac{1}{4}x^4 + 2x^2 + 4$, so $v = \frac{1}{4}x^2 + 1$.

■ **Key knowledge: Choosing the positive velocity branch**

For a force depending on position, use $ma = mv \, dv/dx$. An initial velocity determines the branch after integration.

■ **Worked example: Choosing the positive velocity branch**

A 2 kg particle has force $F = 4x$ N and $v = 3 \text{ m s}^{-1}$ at $x = 0$. Thus $2v \, dv/dx = 4x$. Integrating gives $v^2 = 2x^2 + 9$. For the initially positive motion, $v = \sqrt{2x^2 + 9}$; at $x = 2$, $v = \sqrt{17} \text{ m s}^{-1}$. Squaring has hidden the direction, so restore it deliberately.

2 Practice

2.1 State the form of acceleration to use when a force depends on **position** x . [1]

2.2 A particle has $v \frac{dv}{dx} = 2x$. Find v^2 in terms of x , given $v = 0$ at $x = 0$. [2]

3 Exam-style questions

3.1 For a force that depends on position, acceleration is best written as: [1]

- ☐ A $\frac{dv}{dt}$
☐ B $v \frac{dv}{dx}$
☐ C $\frac{dx}{dt}$
☐ D $\frac{d^2x}{dt^2}$ only

3.2 A particle of mass 2 kg moves along a straight line under a force $F = 6x$ N in the direction of motion. At $x = 0$ the speed is 2 m s^{-1} .

(a) Form a differential equation in v and x . [2]

(b) Find v when $x = 2$. [4]

3.3 A particle of mass 0.5 kg moves so that its acceleration is $\frac{dv}{dt} = 4 - 2t$. At $t = 0$ it is at rest. Find its velocity when $t = 3$ and comment. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $a = v \frac{dv}{dx}$

2.2

- $\int v \, dv = \int 2x \, dx \Rightarrow \frac{1}{2}v^2 = x^2 + C; v = 0 \text{ at } x = 0 \Rightarrow C = 0, \text{ so } v^2 = 2x^2$

3.1 B

- Use $a = v \frac{dv}{dx}$ for a position-dependent force

3.2

(a)

- $2v \frac{dv}{dx} = 6x$, i.e. $v \frac{dv}{dx} = 3x$

(b)

- $\int v \, dv = \int 3x \, dx \Rightarrow \frac{1}{2}v^2 = \frac{3}{2}x^2 + C; \text{ at } x = 0, v = 2 \Rightarrow C = 2; \text{ at } x = 2: \frac{1}{2}v^2 = 6 + 2 = 8 \Rightarrow v^2 = 16; v = 4 \text{ m s}^{-1}$

3.3

- $v = \int (4 - 2t) \, dt = 4t - t^2 + C; v = 0 \text{ at } t = 0 \Rightarrow C = 0; \text{ at } t = 3: v = 12 - 9 = 3 \text{ m s}^{-1};$
the force has reversed (since $4 - 2t < 0$ for $t > 2$), so the particle is slowing after $t = 2$