

2.5 Complex numbers (de Moivre's theorem)

Name: _____ Class: _____ Date: _____

Total: 15 marks

KEY WORDS de Moivre's theorem

• roots of unity

• complex number

Objective

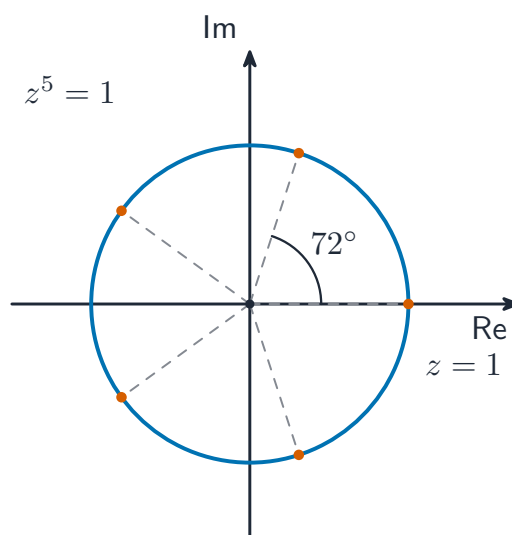
Build the skills to answer exam questions on **complex numbers** – de Moivre's theorem, expanding $\cos n\theta/\sin n\theta$, and the **roots of unity**.

You must be able to:

- use $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$
- express $\cos n\theta/\sin n\theta$ in powers (and powers in multiple angles)
- find the n th roots of a complex number

1 Key knowledge and worked examples

■ De Moivre & roots of unity



The solutions of $z^n = 1$ are equally spaced around the unit circle

$\cos 3\theta$: real part of $(\cos \theta + i \sin \theta)^3 = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = 4 \cos^3 \theta - 3 \cos \theta$.

Roots of unity: $z^n = 1 \Rightarrow z = e^{2\pi ik/n}, k = 0, \dots, n-1$.

■ **Key knowledge: Roots of a non-unit complex number**

Take the n th root of the modulus and include every argument differing by a multiple of 2π . There are n distinct roots for a non-zero number.

■ **Worked example: Roots of a non-unit complex number**

Solve $z^3 = 8i$. Write $8i = 8e^{i(\pi/2+2k\pi)}$. The roots are $z = 2e^{i(\pi/6+2k\pi/3)}$ for $k = 0, 1, 2$, namely $\sqrt{3} + i$, $-\sqrt{3} + i$ and $-2i$. Each has modulus 2 and cubes to $8i$.

2 Practice

2.1 Use de Moivre to write $(\cos \theta + i \sin \theta)^4$ in the form $\cos(\) + i \sin(\)$. [1]

2.2 Write down the three cube roots of unity in polar form. [2]

3 Exam-style questions

3.1 By de Moivre's theorem, $(\cos \theta + i \sin \theta)^5$ equals: [1]

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A $\cos 5\theta + i \sin 5\theta$

B $5 \cos \theta + 5i \sin \theta$

C $\cos \theta + i \sin 5\theta$

D $\cos^5 \theta + i \sin^5 \theta$

3.2 (a) Use de Moivre's theorem to show that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$. [4]

(b) Hence solve $4 \sin^3 \theta - 3 \sin \theta + \frac{1}{2} = 0$ for $0 < \theta < \frac{\pi}{2}$. [3]

3.3 Find the four roots of $z^4 = 16$, giving them in the form $x + iy$. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\cos 4\theta + i \sin 4\theta$

2.2

- $1, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

3.1 A

- $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$

3.2

(a)

- imaginary part of $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$: $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$;
 $= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta$

(b)

- $4 \sin^3 \theta - 3 \sin \theta = -\sin 3\theta$, so the equation is $-\sin 3\theta + \frac{1}{2} = 0 \Rightarrow \sin 3\theta = \frac{1}{2}$; for
 $0 < 3\theta < \frac{3\pi}{2}$, $3\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; $\theta = \frac{\pi}{18}$ or $\frac{5\pi}{18}$

3.3

- $z^4 = 16 \Rightarrow z = 2e^{ik\pi/2}$, $k = 0, 1, 2, 3$; $z = 2, 2i, -2, -2i$