

2.4 Integration (standard forms, reduction formulae)

Name: _____ Class: _____ Date: _____

Total: 16 marks

KEY WORDS completing the square • reduction formula
 • further integration • arc length

Objective

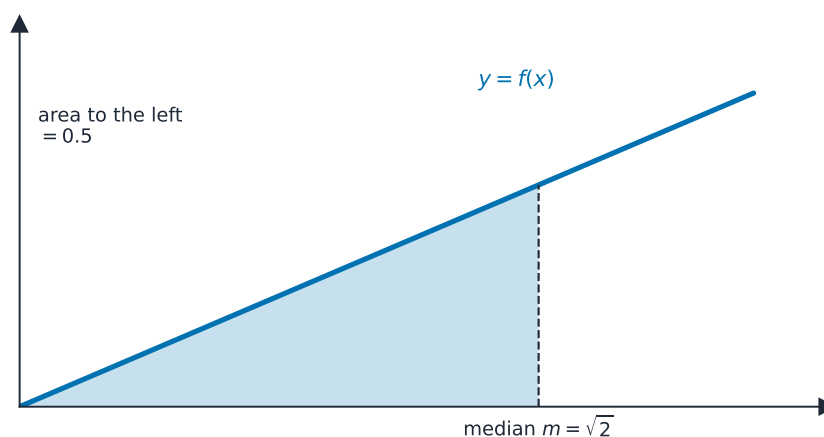
Build the skills to answer exam questions on **further integration** – the standard inverse-function integrals, **reduction formulae**, and **arc length**.

You must be able to:

- use $\int \frac{1}{a^2+x^2} dx$ and $\int \frac{1}{\sqrt{a^2-x^2}} dx$ (and completing the square)
- derive and use a reduction formula
- find an arc length

1 Key knowledge and worked examples

■ Standard integrals & reduction



$$\text{median } m \text{ solves } F(m) = \int_{-\infty}^m f(x) dx = \frac{1}{2}$$

An integral is the area under a curve; the inverse-function forms handle $\frac{1}{a^2+x^2}$ and $\frac{1}{\sqrt{a^2-x^2}}$

$$\int \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1} \frac{x}{2} + C.$$

Reduction: $I_n = \int_0^{\pi/2} \sin^n x \, dx$ satisfies $I_n = \frac{n-1}{n} I_{n-2}$.

■ **Key knowledge: Using a reduction formula**

A reduction formula needs starting values. Reduce the index repeatedly until the integral reaches a known base case.

■ **Worked example: Using a reduction formula**

For $I_n = \int_0^{\pi/2} \sin^n x \, dx$ and $I_n = (n-1)I_{n-2}/n$, use $I_0 = \pi/2$. Then $I_2 = I_0/2 = \pi/4$ and $I_4 = 3I_2/4 = 3\pi/16$. The even-index chain ends at I_0 ; the odd chain ends at $I_1 = 1$.

2 Practice

2.1 Find $\int \frac{1}{9+x^2} dx$. [2]

2.2 Find $\int \frac{1}{\sqrt{25-x^2}} dx$. [2]

3 Exam-style questions

3.1 $\int_0^2 \frac{1}{4+x^2} dx$ equals: [1]

- ☐ **A** $\frac{\pi}{8}$
B $\frac{\pi}{4}$
C $\frac{\pi}{2}$
D π

3.2 Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$.

(a) Using integration by parts, show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$. [4]

(b) Hence evaluate I_4 . [3]

3.3 Find $\int \frac{1}{x^2 - 4x + 13} \, dx$ by completing the square. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\frac{1}{3} \tan^{-1} \frac{x}{3} + C$

2.2

- $\sin^{-1} \frac{x}{5} + C$

3.1 A

- $\left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^2 = \frac{1}{2} \tan^{-1} 1 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$

3.2

(a)

- $I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$; by parts ($u = \sin^{n-1} x$, $dv = \sin x \, dx$): $I_n = [-\sin^{n-1} x \cos x]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx$; the boundary term is 0; $\cos^2 x = 1 - \sin^2 x$ gives $I_n = (n-1)(I_{n-2} - I_n)$; so $nI_n = (n-1)I_{n-2}$, i.e. $I_n = \frac{n-1}{n} I_{n-2}$

(b)

- $I_0 = \frac{\pi}{2}$; $I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}$; $I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}$

3.3

- $x^2 - 4x + 13 = (x-2)^2 + 9$; $\int \frac{1}{(x-2)^2 + 9} \, dx = \frac{1}{3} \tan^{-1} \frac{x-2}{3} + C$