

2.3 Differentiation (Maclaurin, hyperbolic, parametric)

Name: _____ Class: _____ Date: _____

Total: 37 marks

KEY WORDS maclaurin's series

• hyperbolic functions

• further differentiation

Objective

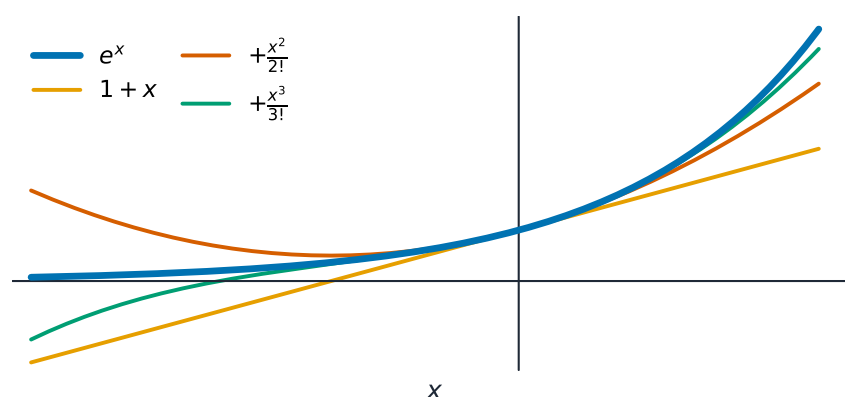
Build the skills to answer exam questions on **further differentiation** – derivatives of inverse and **hyperbolic functions**, second derivatives of implicit/parametric curves, and **Maclaurin's series**.

You must be able to:

- differentiate $\sin^{-1} x$, $\tan^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$
- find $\frac{d^2y}{dx^2}$ for implicit/parametric curves
- find a Maclaurin series for a function

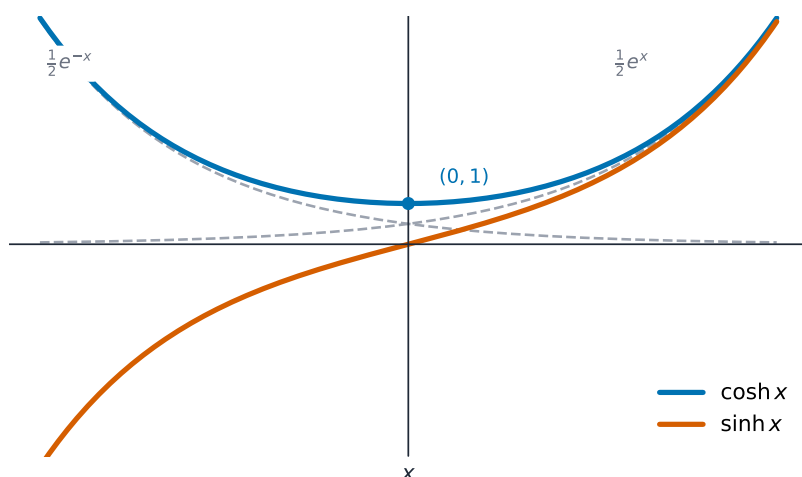
1 Key knowledge and worked examples

■ Maclaurin series



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \cdot \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Adding more terms makes the polynomial match the function over a wider stretch around $x = 0$



$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh^2 x - \sinh^2 x = 1$$

The hyperbolic functions whose derivatives appear here

Maclaurin: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$; e.g. $e^x = 1 + x + \frac{x^2}{2!} + \dots$.

Derivatives: $\frac{d}{dx} \sinh x = \cosh x$; $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1+x^2}$.

■ The derivatives to know, and the substitutions they unlock

Hyperbolic functions, defined from the exponential:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \tanh x = \frac{\sinh x}{\cosh x}$$

The identity is $\cosh^2 x - \sinh^2 x = 1$ – note the **minus**, which is what makes the substitutions differ from the trigonometric ones.

y	$\frac{dy}{dx}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$ (no minus – unlike cos)
$\tanh x$	$\operatorname{sech}^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$

Reading the integral to choose the substitution – the sign under the root decides:

Integrand contains	Substitute	Because
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 = \cos^2$
$a^2 + x^2$	$x = a \tan \theta$	$1 + \tan^2 = \sec^2$
$\sqrt{a^2 + x^2}$	$x = a \sinh u$	$1 + \sinh^2 = \cosh^2$
$\sqrt{x^2 - a^2}$	$x = a \cosh u$	$\cosh^2 - 1 = \sinh^2$

Worked example. Differentiate $y = \operatorname{arsinh}(3x)$ and hence write down $\int \frac{1}{\sqrt{1+9x^2}} dx$.

1. By the chain rule, $\frac{dy}{dx} = \frac{3}{\sqrt{1+(3x)^2}} = \frac{3}{\sqrt{1+9x^2}}$.
2. So $\int \frac{3}{\sqrt{1+9x^2}} dx = \operatorname{arsinh}(3x) + C$.
3. Dividing by 3: $\int \frac{1}{\sqrt{1+9x^2}} dx = \frac{1}{3} \operatorname{arsinh}(3x) + C$.

That is the pattern behind almost every “differentiate, hence integrate” question: the integral you are asked for is the derivative you just found, scaled.

■ Key knowledge: A parametric second derivative

Differentiate the first derivative with respect to the parameter, then divide by dx/dt again. Do not confuse d^2y/dt^2 with d^2y/dx^2 .

■ Worked example: A parametric second derivative

For $x = t^2$, $y = t^3$ and $t > 0$, $dy/dx = (3t^2)/(2t) = 3t/2$. Hence $d^2y/dx^2 = [d(3t/2)/dt]/(2t) = 3/(4t)$. At $t = 2$, this is $3/8$. Writing $y = x^{3/2}$ supplies an independent derivative check.

2 Practice

2.1 Differentiate $y = \cosh(3x)$. [2]

2.2 Differentiate $y = \sin^{-1}(2x)$. [2]

2.3 Differentiate $y = \cosh(4x)$ and $y = \arctan(2x)$. [3]

2.4 State the substitution you would use for $\int \frac{1}{\sqrt{x^2 - 9}} dx$, and why. [2]

2.5 Show that $\cosh^2 x - \sinh^2 x = 1$ using the exponential definitions. [3]

3 Exam-style questions

3.1 $\frac{d}{dx} \tanh x$ equals: [1]

- ☐ **A** $\operatorname{sech}^2 x$
B $\operatorname{cosech}^2 x$
C $\cosh^2 x$
D $\sinh^2 x$

3.2 Find the Maclaurin series for $\ln(1 + x)$ up to and including the term in x^3 . [4]

3.3 A curve is given by $x = t^2$, $y = t^3$. Find $\frac{d^2y}{dx^2}$ in terms of t . [5]

3.4 (a) Differentiate $y = \operatorname{arcosh}(2x)$ with respect to x . [3]

(b) Hence find $\int \frac{1}{\sqrt{4x^2 - 1}} dx$. [2]

(c) Use the substitution $x = 3 \sinh u$ to show that $\int \frac{1}{\sqrt{9 + x^2}} dx = \operatorname{arsinh}\left(\frac{x}{3}\right) + C$. [5]

(d) Differentiate $y = x \arctan x$ and hence evaluate $\int_0^1 \arctan x \, dx$, giving your answer in exact form. [5]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\frac{dy}{dx} = 3 \sinh(3x)$

2.2

- $\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$

2.3

- $\frac{d}{dx} \cosh 4x = 4 \sinh 4x; \frac{d}{dx} \arctan 2x = \frac{2}{1+4x^2}$

2.4

- $x = 3 \cosh u$; because $\cosh^2 u - 1 = \sinh^2 u$ removes the root when $a = 3$

2.5

- $\cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x}); \sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x});$ subtracting,
 $\frac{1}{4}(4) = 1$

3.1 A

- $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

3.2

- $f(0) = 0; f'(x) = \frac{1}{1+x}, f'(0) = 1; f''(x) = -\frac{1}{(1+x)^2}, f''(0) = -1; f'''(0) = 2;$
 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

3.3

- $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2, \text{ so } \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}; \frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{3t}{2} \right) \div \frac{dx}{dt} = \frac{3/2}{2t} = \frac{3}{4t}$

3.4

(a)

- $\frac{dy}{dx} = \frac{2}{\sqrt{(2x)^2 - 1}} = \frac{2}{\sqrt{4x^2 - 1}}$

(b)

- The integrand is half the derivative in

(a)

-, so the integral is $\frac{1}{2} \operatorname{arcosh}(2x) + C$

(c)

- $x = 3 \sinh u$ so $\frac{dx}{du} = 3 \cosh u$; $9 + x^2 = 9(1 + \sinh^2 u) = 9 \cosh^2 u$, so $\sqrt{9 + x^2} = 3 \cosh u$; the integral becomes $\int \frac{3 \cosh u}{3 \cosh u} du = \int 1 du = u + C$; and $u = \operatorname{arsinh}(x/3)$

(d)

- $\frac{d}{dx}(x \arctan x) = \arctan x + \frac{x}{1 + x^2}$; integrating both sides from 0 to 1, $[x \arctan x]_0^1 = \int_0^1 \arctan x dx + [\frac{1}{2} \ln(1 + x^2)]_0^1$; so $\frac{\pi}{4} = I + \frac{1}{2} \ln 2$; $I = \frac{\pi}{4} - \frac{1}{2} \ln 2$