

2.1 Hyperbolic functions

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS hyperbolic functions • logarithmic form

Objective

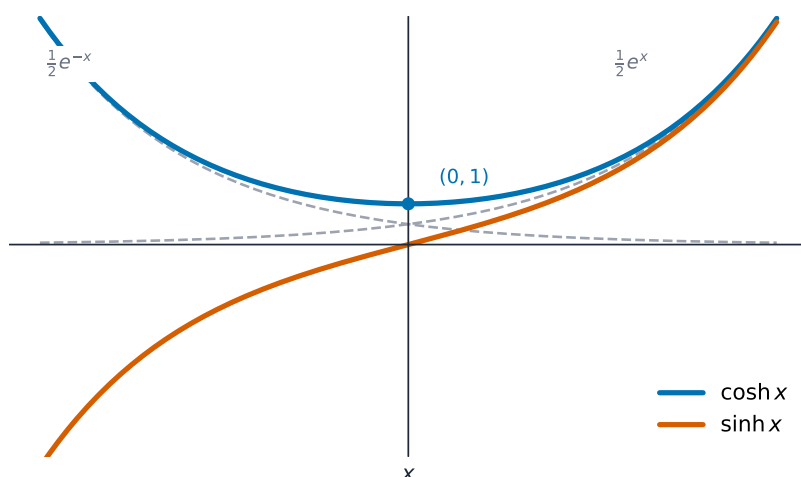
Build the skills to answer exam questions on **hyperbolic functions** – the definitions, identities, and the **logarithmic form** of the inverses.

You must be able to:

- use $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$
- use $\cosh^2 x - \sinh^2 x = 1$ and related identities
- use $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ and solve equations

1 Key knowledge and worked examples

■ Identity & solving



$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh^2 x - \sinh^2 x = 1$$

cosh is the average of e^x and e^{-x} ; sinh is half their difference

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{4}{4} = 1.$$

Solve $\cosh x = 2$: $\frac{e^x + e^{-x}}{2} = 2 \Rightarrow e^{2x} - 4e^x + 1 = 0 \Rightarrow e^x = 2 \pm \sqrt{3}, x = \ln(2 \pm \sqrt{3})$.

■ **Key knowledge: An inverse hyperbolic function**

Derive the logarithmic form from the exponential definition and choose the positive exponential root. This also identifies the valid domain.

■ **Worked example: An inverse hyperbolic function**

For $y = \operatorname{arsinh} x$, $x = (e^y - e^{-y})/2$. Setting $u = e^y > 0$ gives $u^2 - 2xu - 1 = 0$. Thus $u = x + \sqrt{x^2 + 1}$, because the other root is negative. Therefore $y = \ln(x + \sqrt{x^2 + 1})$, valid for every real x .

2 Practice

2.1 Write down the value of $\cosh 0$. [1]

2.2 Evaluate $\sinh^{-1} 0$ using the logarithmic form. [1]

3 Exam-style questions

3.1 The identity satisfied by hyperbolic functions is: [1]

- ☐ **A** $\cosh^2 x + \sinh^2 x = 1$
B $\cosh^2 x - \sinh^2 x = 1$
C $\sinh^2 x - \cosh^2 x = 1$
D $\cosh^2 x - \sinh^2 x = -1$

3.2 Solve the equation $\cosh x = 3$, giving exact answers. [4]

3.3 (a) Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. [4]

(b) Hence evaluate $\sinh^{-1}(\frac{3}{4})$ as a single logarithm. [2]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\cosh 0 = \frac{1}{2}(1 + 1) = 1$

2.2

- $\sinh^{-1} 0 = \ln(0 + \sqrt{0 + 1}) = \ln 1 = 0$

3.1 B

- $\cosh^2 x - \sinh^2 x = 1$

3.2

- $\frac{e^x + e^{-x}}{2} = 3 \Rightarrow e^{2x} - 6e^x + 1 = 0; e^x = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}; x = \ln(3 \pm 2\sqrt{2})$

3.3

(a)

- let $y = \sinh^{-1} x$, so $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$; $e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x + \sqrt{x^2 + 1}$ (positive root); $y = \ln(x + \sqrt{x^2 + 1})$

(b)

- $\sinh^{-1} \frac{3}{4} = \ln \left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right) = \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$