

1.6 Vectors

Name: _____ Class: _____ Date: _____

Total: 18 marks

KEY WORDS plane • vectors • vector product

Objective

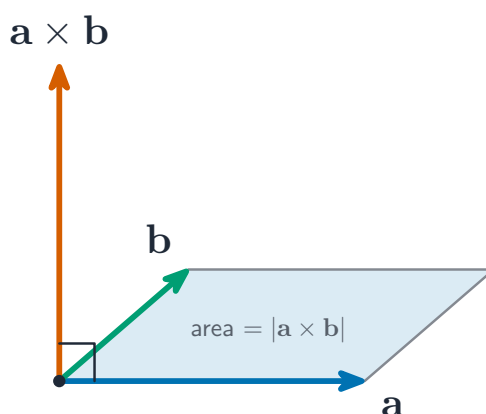
Build the skills to answer exam questions on **vectors** in 3-D – the **vector product**, equations of a **plane**, and distances/angles between lines and planes.

You must be able to:

- compute $\mathbf{a} \times \mathbf{b}$ and use it for a normal/area
- write a plane as $\mathbf{r} \cdot \mathbf{n} = p$ or $ax + by + cz = d$
- find the angle between, and distance from, lines and planes

1 Key knowledge and worked examples

■ Vector product & plane



$\mathbf{a} \times \mathbf{b}$ is perpendicular to both; its length is the area of the parallelogram

$\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$: $\mathbf{a} \times \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

Plane through A with normal $\mathbf{n}$: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

■ **Key knowledge: A point-to-plane distance**

For a plane $\mathbf{n} \cdot \mathbf{r} = d$, perpendicular distance from $\mathbf{p}$ is $|\mathbf{n} \cdot \mathbf{p} - d|/|\mathbf{n}|$. The absolute value makes distance non-negative.

■ **Worked example: A point-to-plane distance**

Find the distance from $(1, 2, 3)$ to $2x - y + 2z = 4$. The normal is $(2, -1, 2)$ with magnitude 3. Therefore $D = |2(1) - 2 + 2(3) - 4|/3 = 2/3$. A point on the plane would make the numerator zero.

2 Practice

2.1 Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$. [3]

2.2 Write the plane through $(1, 2, 3)$ with normal $\mathbf{n} = (1, 1, 1)$ in the form $ax + by + cz = d$. [2]

3 Exam-style questions

3.1 $\mathbf{a} \times \mathbf{b}$ is a vector that is: [1]

- ☐ **A** parallel to $\mathbf{a}$
B perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
C equal to $\mathbf{a} \cdot \mathbf{b}$
D of length $|\mathbf{a}||\mathbf{b}|\cos\theta$

3.2 Points $P(1, 0, 2)$, $Q(3, 1, 1)$ and $R(2, -2, 4)$ are given.

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$, and hence the area of triangle PQR . [4]

(b) Find the equation of the plane PQR , and the perpendicular distance from the origin to it. [4]

3.3 Find the acute angle between the planes $x + 2y + 2z = 5$ and $2x - y + 2z = 1$. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

$$\bullet \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(2) + \mathbf{k}(-1) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

2.2

$$\bullet x + y + z = 1 + 2 + 3 = 6$$

3.1 B

- The vector product is perpendicular to both vectors

3.2

(a)

$$\bullet \overrightarrow{PQ} = (2, 1, -1), \overrightarrow{PR} = (1, -2, 2); \text{ cross} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = \mathbf{i}(2 - 2) - \mathbf{j}(4 + 1) + \mathbf{k}(-4 - 1) = (0, -5, -5); \text{ the triangle is half the parallelogram, so its area is } \frac{1}{2} |(0, -5, -5)| = \frac{1}{2} \cdot 5\sqrt{2} = \frac{5\sqrt{2}}{2}$$

(b)

- any parallel normal will do, so take $(0, 1, 1)$; through $P(1, 0, 2)$ this gives $y + z = 2$.
Distance from the origin $= \frac{|0 + 0 - 2|}{\sqrt{0^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

3.3

- normals $(1, 2, 2)$ and $(2, -1, 2)$; dot $= 2 - 2 + 4 = 4$; $|\mathbf{n}_1| = 3$, $|\mathbf{n}_2| = 3$; $\cos \theta = \frac{4}{9} \Rightarrow \theta = 63.6^\circ$