

1.3 Summation of series

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS method of differences

• partial fractions

• sum to infinity

• summation of series

Objective

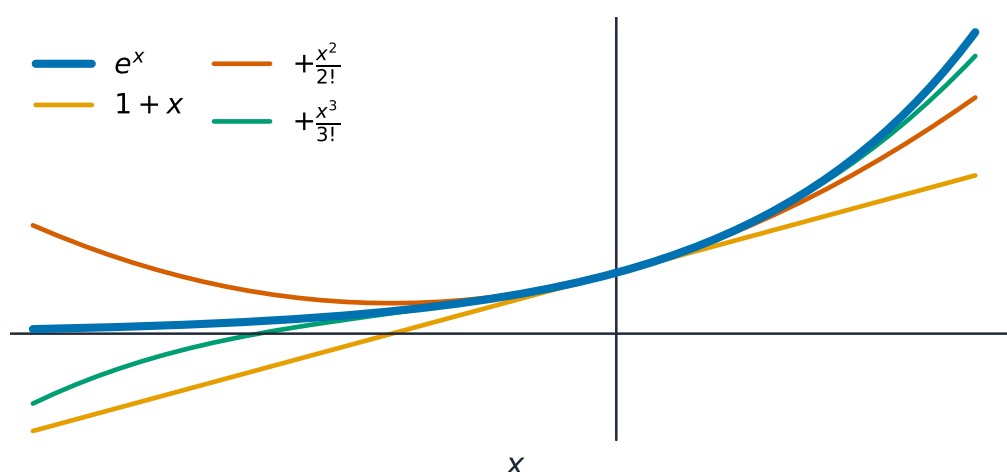
Build the skills to answer exam questions on the **summation of series** – the standard results for $\sum r$, $\sum r^2$, $\sum r^3$, and the **method of differences**.

You must be able to:

- use $\sum r = \frac{1}{2}n(n+1)$, $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum r^3 = \frac{1}{4}n^2(n+1)^2$
- sum a series by the method of differences (telescoping)
- decide convergence and find a sum to infinity

1 Key knowledge and worked examples

■ Standard results & differences



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \cdot \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

A finite sum builds from standard results; telescoping cancels all but the end terms

$$\sum_{r=1}^n (2r+1) = 2 \cdot \frac{1}{2}n(n+1) + n = n(n+2).$$

Differences: if $u_r = f(r) - f(r+1)$ then $\sum_{r=1}^n u_r = f(1) - f(n+1)$.

■ Key knowledge: A telescoping sum

Write each term as a difference, display the first and last few terms, and retain the uncanceled boundary terms.

■ Worked example: A telescoping sum

Find $S_n = \sum_{r=1}^n 1/[r(r+1)]$. Since $1/[r(r+1)] = 1/r - 1/(r+1)$, $S_n = (1 - 1/2) + (1/2 - 1/3) + \cdots + (1/n - 1/(n+1)) = 1 - 1/(n+1)$. Therefore $S_n = n/(n+1)$ and its limit is 1.

2 Practice

2.1 Evaluate $\sum_{r=1}^n r^2$ as a formula in n . [1]

2.2 Find $\sum_{r=1}^{10} (3r - 1)$. [2]

3 Exam-style questions

3.1 $\sum_{r=1}^n r^3$ equals: [1]

- ☐ **A** $\frac{1}{4}n^2(n+1)^2$
B $\frac{1}{6}n(n+1)(2n+1)$
C $\frac{1}{2}n(n+1)$
D n^3

3.2 (a) Express $\frac{1}{r(r+1)}$ in partial fractions. [2]

(b) Hence find $\sum_{r=1}^n \frac{1}{r(r+1)}$ by the method of differences, and state its sum to infinity. [4]

3.3 Find $\sum_{r=1}^n (r^2 + r)$, giving your answer as a single factorised expression. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\frac{1}{6}n(n+1)(2n+1)$

2.2

- $3 \cdot \frac{1}{2}(10)(11) - 10 = 165 - 10 = 155$

3.1 A

- $\sum r^3 = \frac{1}{4}n^2(n+1)^2$

3.2

(a)

- $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

(b)

- $\sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$; as $n \rightarrow \infty$ the sum $\rightarrow 1$

3.3

- $\sum (r^2 + r) = \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1); = \frac{1}{6}n(n+1)[(2n+1) + 3]; = \frac{1}{6}n(n+1)(2n+4) = \frac{1}{3}n(n+1)(n+2)$