

1.2 Rational functions and graphs

Name: _____ Class: _____ Date: _____

Total: 15 marks

KEY WORDS asymptotes • oblique • discriminant
 • rational function • rational functions and graphs

Objective

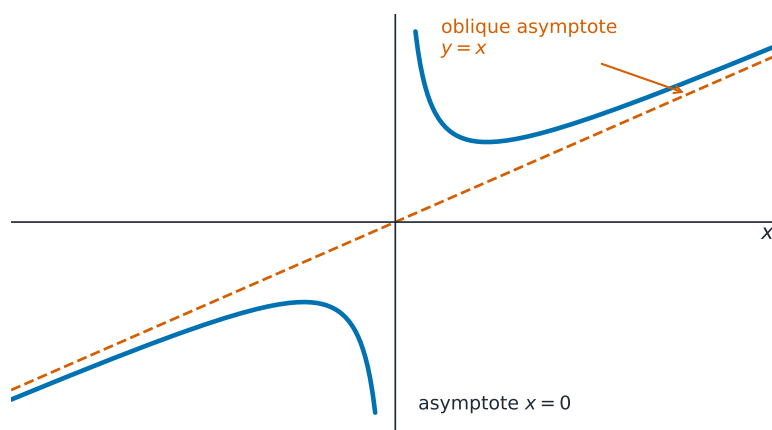
Build the skills to answer exam questions on **rational functions and graphs** – **asymptotes** (including **oblique**), the range via the **discriminant**, and related graphs.

You must be able to:

- find vertical and oblique asymptotes (divide out)
- find the set of values of a rational function using the discriminant
- relate $y = f(x)$ to $y = \frac{1}{f(x)}$, $y = |f(x)|$, $y^2 = f(x)$

1 Key knowledge and worked examples

■ Asymptotes & range



oblique asymptote when $\deg(\text{num}) = \deg(\text{den}) + 1$ (long division)

Far out the curve hugs the slant line; near a zero of the denominator it runs off along a vertical asymptote

$y = \frac{x^2 + 1}{x} = x + \frac{1}{x}$: vertical asymptote $x = 0$, oblique asymptote $y = x$.

Range of $y = \frac{x^2 + 1}{x}$: $yx = x^2 + 1 \Rightarrow x^2 - yx + 1 = 0$; real x needs $y^2 - 4 \geq 0$, so $y \leq -2$ or $y \geq 2$.

■ **Key knowledge: An oblique asymptote**

Polynomial division separates a rational function into its asymptotic line and a term tending to zero. Check for cancelled factors before declaring a vertical asymptote.

■ **Worked example: An oblique asymptote**

For $y = (x^2 + 1)/(x - 1)$, division gives $y = x + 1 + 2/(x - 1)$. Thus the oblique asymptote is $y = x + 1$ and the vertical asymptote is $x = 1$. The remainder is non-zero, so there is no cancellation at $x = 1$.

2 Practice

2.1 Write down the vertical asymptote of $y = \frac{2x + 1}{x - 3}$. [1]

2.2 Find the oblique asymptote of $y = \frac{x^2}{x - 1}$. [2]

3 Exam-style questions

3.1 The curve $y = \frac{x^2 - 4}{x}$ has an oblique asymptote: [1]

- ☐ **A** $y = x$
B $y = x - 4$
C $y = 2x$
D $y = -4$

3.2 A curve has equation $y = \frac{x^2 + 2}{x - 1}$.

(a) Find the equations of the two asymptotes. [3]

(b) Use the discriminant to find the set of values that y **cannot** take. [4]

3.3 The function is $y = \frac{x}{x^2 + 1}$. Find the set of values y can take. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $x = 3$

2.2

- $\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$, so oblique asymptote $y = x + 1$

3.1 A

- $\frac{x^2 - 4}{x} = x - \frac{4}{x}$, so the oblique asymptote is $y = x$

3.2

(a)

- vertical $x = 1$; divide: $\frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}$, oblique $y = x + 1$

(b)

- $y(x-1) = x^2+2 \Rightarrow x^2-yx+(y+2) = 0$; real x needs $y^2-4(y+2) \geq 0 \Rightarrow y^2-4y-8 \geq 0$; roots $y = 2 \pm 2\sqrt{3}$; so y cannot take values in $2 - 2\sqrt{3} < y < 2 + 2\sqrt{3}$

3.3

- $y(x^2 + 1) = x \Rightarrow yx^2 - x + y = 0$; for real x (with $y \neq 0$), $1 - 4y^2 \geq 0$; $-\frac{1}{2} \leq y \leq \frac{1}{2}$