

Exercise walk-through

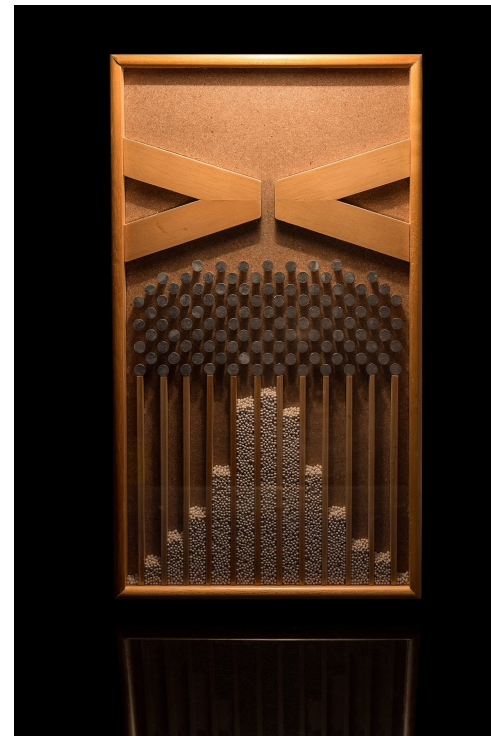
4.5

Probability generating functions

A-Level Further Mathematics

What we will cover

- 1 Continuous random variables
- 2 t -distribution inference
- 3 Chi-squared tests
- 4 Non-parametric tests
- 5 Generating functions
- 6 Recap



Worked example – a sign test

Ten values are tested against H_0 : median = 5. Nine fall above it. Is that evidence against H_0 ?

- under H_0 , each value is above with probability 0.5, so the count $X \sim B(10, 0.5)$
- $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + \binom{10}{10}(0.5)^{10}$
- $= \frac{10 + 1}{1024} = 0.0107$
- $0.0107 < 0.05$, so **reject** H_0 at the 5% level

The whole test is a binomial tail. Note it uses *only* the signs – discard how far above the median each value was, and a great deal of information with it.

Which test, and on what assumption

Hypothesis test

Every **hypothesis test** states H_0 , a test statistic, a significance level and a conclusion in context.

Parametric tests

t and χ^2 assume a **theoretical distribution** – normality, or expected counts of at least 5.

Wilcoxon signed-rank

The **Wilcoxon signed-rank test** : paired data, no normality assumed. Rank the differences and compare the rank sums.

Wilcoxon rank-sum

The **Wilcoxon rank-sum test** : two *independent* samples, again distribution-free.

A **probability density function** is what the parametric tests assume and the non-parametric ones avoid – which is why the latter are used on small or skewed samples.

Exercises 2.1·2.2

2.1 Write down $G(t)$ for a variable with $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$. [2]

2.2 State the formula for $E(X)$ in terms of G . [1]

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- $G(t) = \frac{1}{3}(t + t^2 + t^3)$

2.2 State the formula for $E(X)$ in terms of G . [1]

2.2

- $E(X) = G'(1)$

Exercise 3.1

3.1 For any PGF, $G(1)$ equals:

[1]

☐

A 0

B $E(X)$

C 1

D $\text{Var}(X)$

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3.1 C

- $G(1) = \sum P(X = x) = 1$

Exercise 3.2 (a)

3.2 A variable has $G(t) = \frac{1}{4}(1+t)^2$.

(a) Find the probability distribution of X .

[3]

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(a)

- $G(t) = \frac{1}{4}(1 + 2t + t^2) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$, so $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$

Exercise 3.2 (b)

3.2 A variable has $G(t) = \frac{1}{4}(1+t)^2$.

(b) Find $E(X)$ and $\text{Var}(X)$.

[4]

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(b)

- $G'(t) = \frac{1}{2} + \frac{1}{2}t \Rightarrow E(X) = G'(1) = 1; G''(t) = \frac{1}{2} \Rightarrow \text{Var}(X) = \frac{1}{2} + 1 - 1^2 = \frac{1}{2}$

Exercise 3.3

3.3 X and Y are independent with PGFs $G_X(t) = \frac{1}{2}(1 + t)$ and $G_Y(t) = \frac{1}{3}(1 + t + t^2)$. Find the PGF of $X + Y$ and hence $P(X + Y = 0)$. [3]

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- $G_{X+Y}(t) = G_X(t)G_Y(t) = \frac{1}{2}(1+t) \cdot \frac{1}{3}(1+t+t^2) = \frac{1}{6}(1+t)(1+t+t^2)$; $P(X+Y=0)$ is the constant term $= \frac{1}{6}$