

Exercise walk-through

4.1

Continuous random variables

A-Level Further Mathematics

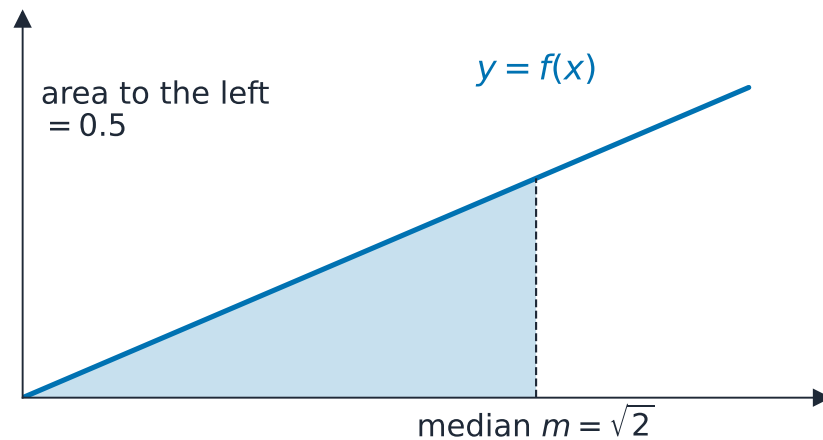
Continuous random variables

$$F(x) = P(X \leq x) = \int_{-\infty}^x f, \text{ and } f = F'.$$

$$f(x) = \frac{1}{2}x, 0 \leq x \leq 2$$

$$F = \frac{1}{4}x^2; \text{ median } m = \sqrt{2} = 1.41$$

Continuous random variables, in detail



$$\text{median } m \text{ solves } F(m) = \int_{-\infty}^m f(x) dx = \frac{1}{2}$$

median: area 0.5 to its left

$$F(x) = P(X \leq x) = \int_{-\infty}^x f, \text{ and } f = F'.$$

The cumulative distribution function

cdf

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt - \text{the running total of the density}$$

going back

$$f(x) = \frac{dF}{dx}: \text{differentiate the cdf to recover the density}$$

its properties

$$\text{non-decreasing, } F(-\infty) = 0, F(\infty) = 1$$

the median

$$\text{solve } F(m) = 0.5; \text{ percentiles}$$

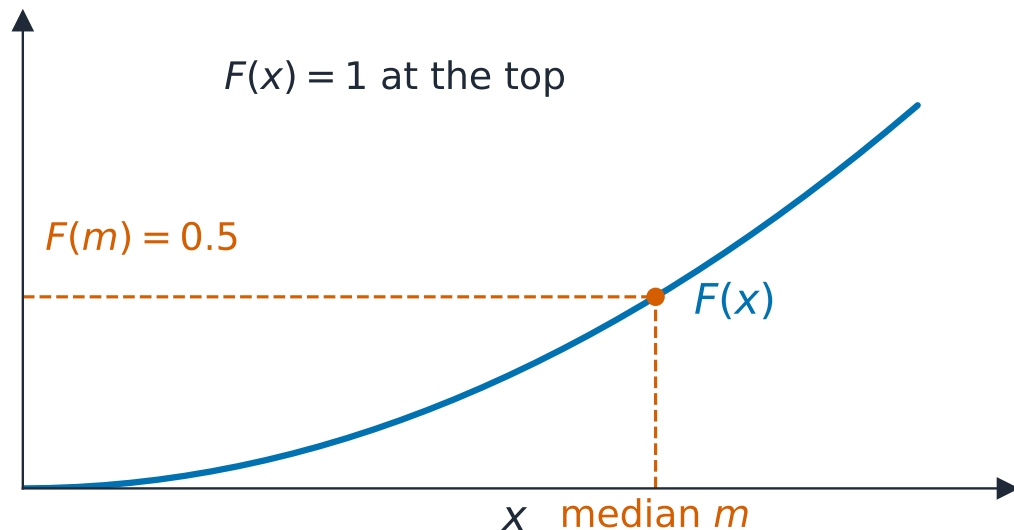
$$\text{solve } F(x) = p$$

why it matters

$$\text{any probability is a difference of two values: } P(a < X \leq b) = F(b) - F(a)$$

For a probability, reach for F ; for a mode or an expectation, reach for f . Knowing which one the question needs saves the whole calculation.

The cumulative graph $F(x)$ rises from 0



$$F(x) = P(X \leq x) = \int_{-\infty}^x f \cdot 0 \leq F \leq 1 \cdot \text{median: } F(m) = \frac{1}{2}$$

The cumulative graph $F(x)$ rises from 0 to 1; the height 0.5 is reached at the median.

The distribution of a related variable

the setup

X has a known distribution and $Y = g(X)$; find Y 's

the method

work through the **cumulative** function, never the density directly

for $Y = X^2$

$$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

then

differentiate to get $f_Y(y)$

the caution

mind the **range** of Y , and whether g is one-to-one over X 's range

Never substitute into the density and hope. The cdf route is the only one that handles a many-to-one g correctly.

Exercises 2.1·2.2

2.1 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $E(X)$. [2]

2.2 Write the relationship between $f(x)$ and $F(x)$. [1]

Exercises 2.1·2.2

2.1 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $E(X)$. [2]

2.1

- $$E(X) = \int_0^2 x \cdot \frac{1}{2}x \, dx = \left[\frac{x^3}{6} \right]_0^2 = \frac{4}{3}$$

2.2 Write the relationship between $f(x)$ and $F(x)$. [1]

2.2

- $$f(x) = F'(x)$$

Exercise 3.1

3.1 The median of a continuous variable solves:

[1]

☐

A $f(m) = 0.5$

B $F(m) = 0.5$

C $E(X) = m$

D $F(m) = 1$

Exercise 3.1

3.1 The median of a continuous variable solves:

[1]

☐

A $f(m) = 0.5$

B $F(m) = 0.5$

C $E(X) = m$

D $F(m) = 1$

3.1 B

- The median solves $F(m) = 0.5$

Exercise 3.2 (a)

3.2 A continuous variable has $f(x) = kx^2$ for $0 \leq x \leq 3$ (and 0 elsewhere).

(a) Find k .

[3]

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(a) Find k .

[3]

(a)

- $$\int_0^3 kx^2 dx = \left[\frac{kx^3}{3} \right]_0^3 = 9k = 1 \Rightarrow k = \frac{1}{9}$$

Exercise 3.2 (b)

3.2 A continuous variable has $f(x) = kx^2$ for $0 \leq x \leq 3$ (and 0 elsewhere).

(b) Find $E(X)$.

[3]

Exercise 3.2 (b)

3.2 A continuous variable has $f(x) = kx^2$ for $0 \leq x \leq 3$ (and 0 elsewhere).

(b) Find $E(X)$.

[3]

(b)

$$\bullet \quad E(X) = \int_0^3 x \cdot \frac{1}{9}x^2 dx = \frac{1}{9} \left[\frac{x^4}{4} \right]_0^3 = \frac{1}{9} \cdot \frac{81}{4} = \frac{9}{4}$$

Exercise 3.3 (a)

3.3 A variable has cdf $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$.

(a) Find the pdf $f(x)$.

[2]

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3.3 A variable has cdf $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$.

(a) Find the pdf $f(x)$.

[2]

(a)

- $f(x) = F'(x) = \frac{x}{8}$ for $0 \leq x \leq 4$

Exercise 3.3 (b)

3.3 A variable has cdf $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$.

(b) Find the median.

[2]

Exercise 3.3 (b)

3.3 A variable has cdf $F(x) = \frac{x^2}{16}$ for $0 \leq x \leq 4$.

(b) Find the median.

[2]

(b)

- $\frac{m^2}{16} = 0.5 \Rightarrow m^2 = 8 \Rightarrow m = 2\sqrt{2} = 2.83$