Exercise walk-through

3.5
Linear motion under a variable force

A-Level Further Mathematics

Why a fast object slows differently

Start with the force model

A bicycle usually feels more air resistance when it moves faster. Its **drag** may depend on **speed** . A different force might depend on **position** or time; these are different models.

A **variable force** can produce changing acceleration. Start with Newton's law and set up a **differential equation** for the motion; constant-acceleration formulae need not apply.

Choose a positive direction first

Start with the force model

Take displacement x positive to the right, and write **velocity** $v = dx/dt$. It can be negative. Speed is $|v|$, so a negative velocity does not mean a negative speed.

Add signed forces on the particle: $m\,dv/dt = \sum F$. For motion to the right, resistance is negative. After reversal, ordinary resistance reverses too; a prescribed force towards a fixed point follows its own direction rule.

Change the variable using the chain rule

Start with the force model

If velocity is written as a function of position along a branch of the motion, the **chain rule** gives

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

This is useful for finding v at a position without first finding time. At a **turning point** $v(x)$ may have a vertical tangent; do not assume $a = 0$ merely because $v = 0$. Use the original force law there.

Choose the variables for the requested answer

Start with the force model

For a time-dependent force, $m\,dv/dt = F(t)$ often gives $v(t)$ directly. For a position-dependent force, $mv\,dv/dx = F(x)$ often gives $v(x)$ directly.

If force depends on velocity, either form may help: use dv/dt for a time question and $v\,dv/dx$ for a distance question. These are choices of variables for the same Newton equation, not different laws.

Choose acceleration variables to match the force model and requested result

Start with signed Newton equation: ma = ΣF

Force uses time	Force uses position	Force uses velocity
↓	↓	↓
$a = dv/dt$	$a = v\,dv/dx$	Either form
Find v(t), then x(t)	Find v(x), choose its sign	Choose time or distance

Separate → integrate → use the initial state
Check force signs and the valid branch.

Choose acceleration variables to match the force model and requested result

Exercises 2.1·2.2

2.1 State the form of acceleration to use when a force depends on **position** x. [1]

2.2 A particle has $v\dfrac{dv}{dx} = 2x$. Find v^2 in terms of x, given $v = 0$ at $x = 0$. [2]

Exercises 2.1·2.2

2.1 State the form of acceleration to use when a force depends on **position** x. [1]

2.1

- $a = v\dfrac{dv}{dx}$

2.2 A particle has $v\dfrac{dv}{dx} = 2x$. Find v^2 in terms of x, given $v = 0$ at $x = 0$. [2]

2.2

- $\int v\,dv = \int 2x\,dx \Rightarrow \frac{1}{2}v^2 = x^2 + C$; $v = 0$ at $x = 0 \Rightarrow C = 0$, so $v^2 = 2x^2$

Exercise 3.1

3.1 For a force that depends on position, acceleration is best written as: [1]

- ☐ **A** $\dfrac{dv}{dt}$
- **B** $v\dfrac{dv}{dx}$
- **C** $\dfrac{dx}{dt}$
- **D** $\dfrac{d^2x}{dt^2}$ only

Exercise 3.1

3.1 For a force that depends on position, acceleration is best written as: [1]

- ☐ **A** $\dfrac{dv}{dt}$
- **B** $v\dfrac{dv}{dx}$
- **C** $\dfrac{dx}{dt}$
- **D** $\dfrac{d^2x}{dt^2}$ only

3.1 B

- Use $a = v\dfrac{dv}{dx}$ for a position-dependent force

Exercise 3.2 (a)

3.2 A particle of mass 2 kg moves along a straight line under a force $F = 6x$ N in the direction of motion. At $x = 0$ the speed is 2 m s^{-1}.

(a) Form a differential equation in v and x. [2]

Exercise 3.2 (a)

3.2 A particle of mass 2 kg moves along a straight line under a force $F = 6x$ N in the direction of motion. At $x = 0$ the speed is 2 m s^{-1}.

(a) Form a differential equation in v and x. [2]

(a)

- $2v\dfrac{dv}{dx} = 6x$, i.e. $v\dfrac{dv}{dx} = 3x$

Exercise 3.2 (b)

3.2 A particle of mass 2 kg moves along a straight line under a force $F = 6x$ N in the direction of motion. At $x = 0$ the speed is 2 m s^{-1}.

(b) Find v when $x = 2$. [4]

Exercise 3.2 (b)

3.2 A particle of mass 2 kg moves along a straight line under a force $F = 6x$ N in the direction of motion. At $x = 0$ the speed is 2 m s^{-1}.

(b) Find v when $x = 2$. [4]

(b)

- $\int v\,dv = \int 3x\,dx \Rightarrow \frac{1}{2}v^2 = \frac{3}{2}x^2 + C$; at $x = 0$, $v = 2 \Rightarrow C = 2$; at $x = 2$: $\frac{1}{2}v^2 = 6 + 2 = 8 \Rightarrow v^2 = 16$; $v = 4$ m s^{-1}

Exercise 3.3

3.3 A particle of mass 0.5 kg moves so that its acceleration is $\dfrac{dv}{dt} = 4 - 2t$. At $t = 0$ it is at rest. Find its velocity when $t = 3$ and comment. [4]

Exercise 3.3

3.3 A particle of mass 0.5 kg moves so that its acceleration is $\dfrac{dv}{dt} = 4 - 2t$. At $t = 0$ it is at rest. Find its velocity when $t = 3$ and comment. [4]

3.3

- $v = \int (4-2t)\,dt = 4t - t^2 + C$; $v = 0$ at $t = 0 \Rightarrow C = 0$; at $t = 3$: $v = 12 - 9 = 3$ m s^{-1}; the force has reversed (since $4 - 2t < 0$ for $t > 2$), so the particle is slowing after $t = 2$