Exercise walk-through

2.1 Hyperbolic functions

A-Level Further Mathematics

A hanging cable needs a hyperbolic curve

A flexible, uniform chain hanging under gravity forms a **catenary** . A shifted and scaled model is

$$y = a\cosh((x-h)/a) + k, \qquad a > 0.$$

Its lowest point is $(h, a+k)$: the input is zero at $x = h$ and $\cosh 0 = 1$. The model assumes uniform weight per unit length, flexibility and no other loads.

> Start with the model's assumptions. A hanging chain is not generally a parabola.

Define all six functions from exponentials

$$\sinh x = (e^x - e^{-x})/2, \qquad \cosh x = (e^x + e^{-x})/2,$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}},$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Since $e^x > 0$, cosh never vanishes. Sinh, an odd function, vanishes only at $x = 0$.

> A reciprocal is not an inverse function. Cosech and coth exclude zero.

Worked: exact values at a logarithm

Set $x = \ln 2$. Then $e^x = 2$ and $e^{-x} = 1/2$.

$$\begin{aligned}\sinh x &= (2 - 1/2)/2 = 3/4,\\ \cosh x &= (2 + 1/2)/2 = 5/4,\\ \tanh x &= (3/4)/(5/4) = 3/5.\end{aligned}$$

The reciprocals give $\operatorname{cosech} x = 4/3$, $\operatorname{sech} x = 4/5$ and $\coth x = 5/3$.

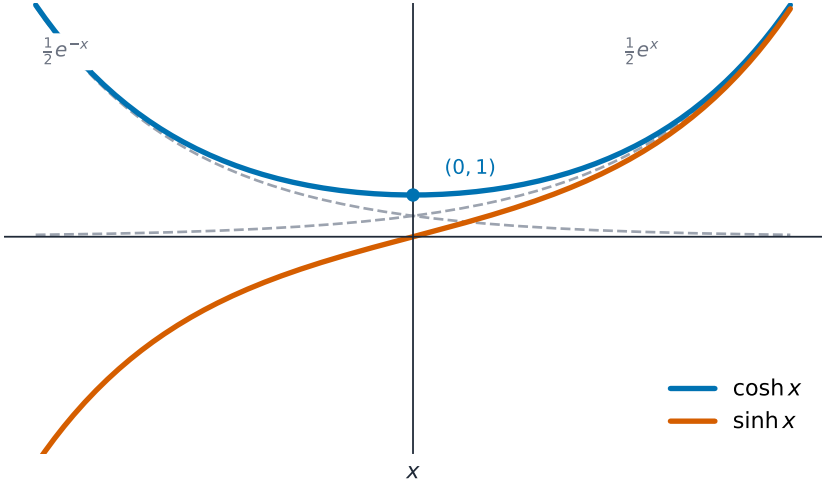
> Check $(5/4)^2 - (3/4)^2 = 1$ before reusing these values.

Record all six domains and ranges

function	domain	range
$\sinh x$	$\mathbb{R}$	$\mathbb{R}$
$\cosh x$	$\mathbb{R}$	$[1, \infty)$
$\tanh x$	$\mathbb{R}$	$(-1, 1)$
$\operatorname{sech} x$	$\mathbb{R}$	$(0, 1]$
$\operatorname{cosech} x$	$\mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$
$\coth x$	$\mathbb{R} \setminus \{0\}$	$(-\infty, -1) \cup (1, \infty)$

Cosh and sech are even; the other four are odd. Denominator zeros exclude inputs, and limiting values need not be attained.

Basic graphs: symmetry and limiting values



$\cosh x = \frac{e^x + e^{-x}}{2}$ · $\sinh x = \frac{e^x - e^{-x}}{2}$ · $\cosh^2 x - \sinh^2 x = 1$

Cosh is even with minimum $(0, 1)$; sinh is odd and passes through the origin.

Exercises 2.1·2.2

2.1 Write down the value of $\cosh 0$. [1]

2.2 Evaluate $\sinh^{-1} 0$ using the logarithmic form. [1]

Exercises 2.1·2.2

2.1 Write down the value of $\cosh 0$. [1]

2.1

- $\cosh 0 = \frac{1}{2}(1+1) = 1$

2.2 Evaluate $\sinh^{-1} 0$ using the logarithmic form. [1]

2.2

- $\sinh^{-1} 0 = \ln(0 + \sqrt{0+1}) = \ln 1 = 0$

Exercises 3.1·3.2

3.1 The identity satisfied by hyperbolic functions is: [1]

☐ **A** $\cosh^2 x + \sinh^2 x = 1$
B $\cosh^2 x - \sinh^2 x = 1$
C $\sinh^2 x - \cosh^2 x = 1$
D $\cosh^2 x - \sinh^2 x = -1$

3.2 Solve the equation $\cosh x = 3$, giving exact answers. [4]

Exercises 3.1·3.2

3.1 The identity satisfied by hyperbolic functions is: [1]

- ☐ **A** $\cosh^2 x + \sinh^2 x = 1$
- **B** $\cosh^2 x - \sinh^2 x = 1$
- **C** $\sinh^2 x - \cosh^2 x = 1$
- **D** $\cosh^2 x - \sinh^2 x = -1$

3.1 B

- $\cosh^2 x - \sinh^2 x = 1$

3.2 Solve the equation $\cosh x = 3$, giving exact answers. [4]

3.2

- $\dfrac{e^x + e^{-x}}{2} = 3 \Rightarrow e^{2x} - 6e^x + 1 = 0;\ e^x = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2};\ x = \ln(3 \pm 2\sqrt{2})$

Exercise 3.3

3.3 (a) Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. [4]

(b) Hence evaluate $\sinh^{-1}(\frac{3}{4})$ as a single logarithm. [2]

Exercise 3.3

3.3 (a) Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. [4]

(b) Hence evaluate $\sinh^{-1}(\frac{3}{4})$ as a single logarithm. [2]

3.3

(a)

- let $y = \sinh^{-1} x$, so $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$; $e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x + \sqrt{x^2 + 1}$ (positive root); $y = \ln(x + \sqrt{x^2 + 1})$

(b)

- $\sinh^{-1} \frac{3}{4} = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$