

Exercise walk-through

1.4

Matrices

A-Level Further Mathematics

Check the sizes before calculating

An $m \times n$ matrix times an $n \times p$ matrix gives an $m \times p$ matrix. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

Multiply each row by each column:

$$AB = \begin{pmatrix} 1 & -4 \\ 9 & 1 \end{pmatrix}, \quad (AB)_{12} = 1(0) + 2(-1) + (-1)(2) = -4.$$

BA exists but is 3×3 ; it cannot equal AB .

Addition/subtraction need matching sizes. A zero matrix has only 0s; for A of size $m \times n$, $I_m A = A = A I_n$.

A determinant tests whether an inverse exists

For a 2×2 matrix,

$$T^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad - bc \neq 0.$$

For a 3×3 example, expand with positional signs $+, -, +$:

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \det M = 1(1) - 2(-2) = 5.$$

A square matrix with zero determinant is **singular**. It has no inverse. A singular plane map may collapse to a line *or to a point*.

For a parameter-dependent determinant, identify every zero before dividing.

Worked: find every singular parameter

Consider $M_a = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ a & 0 & 1 \end{pmatrix}$. Expansion across the first row gives

$$\det M_a = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ a & 1 \end{vmatrix} = 1 + 2a.$$

It is singular exactly when $a = -1/2$. For every other a , the cofactor-transpose method gives an inverse with denominator $1 + 2a$.

Establish the allowed parameter values before dividing by the determinant.

Worked: cofactor, transpose, divide

For the matrix M with determinant 5, form the **cofactor** matrix using the sign pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$:

$$C = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{5} C^T = \frac{1}{5} \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 4 & 1 \end{pmatrix}.$$

Check by multiplication. For instance, entry $(2, 1)$ is

$$(MM^{-1})_{21} = \frac{0(1) + 1(2) + 1(-2)}{5} = 0.$$

Check the other eight entries against I_3 too.

A matrix of cofactors is not yet the inverse: transpose it and divide by the determinant.

Undo operations in reverse order

For non-singular square matrices of the same size,

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

Why? Associativity allows adjacent inverse pairs to cancel:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I.$$

From $AXB = C$, multiply on the correct sides:

$$A^{-1}(AXB)B^{-1} = A^{-1}CB^{-1} \implies X = A^{-1}CB^{-1}.$$

Do not move a matrix through another matrix. Multiplication is generally non-commutative.

Exercises 2.1·2.2

2.1 Find $\det \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$. [1]

2.2 Find the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. [2]

Exercises 2.1·2.2

2.1 Find $\det \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$. [1]

2.1

- $\det = 5(4) - 2(3) = 14$

2.2 Find the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. [2]

2.2

- $\det = 2 - 1 = 1$; inverse = $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Exercise 3.1

3.1 The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents:

[1]

☐

A a rotation

B an enlargement, scale factor 2

C a reflection

D a shear

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3.1 B

- A scalar multiple of I is an enlargement; here scale factor 2

Exercise 3.2 (a)

3.2 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

(a) Find AB .

[2]

Exercise 3.2 (a)

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(a) Find AB .

[2]

(a)

- $AB = \begin{pmatrix} 1 \cdot 2 + 2 \cdot 1 & 1 \cdot 0 + 2 \cdot 3 \\ 3 \cdot 2 + 4 \cdot 1 & 3 \cdot 0 + 4 \cdot 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 10 & 12 \end{pmatrix}$

Exercise 3.2 (b)

3.2 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

(b) Find $\det(AB)$ and verify it equals $\det A \times \det B$.

[3]

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(b) Find $\det(AB)$ and verify it equals $\det A \times \det B$.

[3]

(b)

- $\det(AB) = 48 - 60 = -12$; $\det A = -2$, $\det B = 6$, product = -12 ✓

Exercise 3.3 (a)

3.3 The transformation T has matrix $M = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find the area scale factor of T .

[1]

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(a) Find the area scale factor of T .

[1]

(a)

- area scale factor = $\det M = 6$

Exercise 3.3 (b)

3.3 The transformation T has matrix $M = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) Find M^{-1} and hence the image-to-object matrix that undoes T .

[3]

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3.3 The transformation T has matrix $M = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) Find M^{-1} and hence the image-to-object matrix that undoes T . [3]

(b)

- $M^{-1} = \frac{1}{6} \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$; this matrix undoes T