A-LEVEL COMPUTER SCIENCE • EXERCISE SHEET

13.3.1 The mantissa-exponent trade-off, normalisation, and overflow and underflow

Name: _____ Class: _____ Date: _____ **Total: 26 marks**

KEY WORDS mantissa 尾数 • exponent 指数 • floating-point 浮点数 • overflow 溢出 • underflow 下溢

Objective

Build the skills to answer the **floating-point** 浮点数 questions that sheet 13.3 does not reach. That sheet describes the format, converts both ways and normalises. Two things the syllabus names never appear in it: the effect of **changing the allocation of bits** between mantissa and exponent, and **overflow** 溢出 and **underflow** 下溢. The words allocation, range, overflow and underflow occur in it zero times each.

You must be able to:

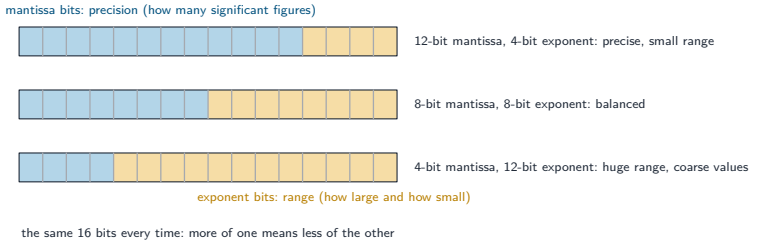
- understand the effects of changing the allocation of bits to **mantissa** 尾数 and **exponent** 指数 in a floating-point representation
- normalise a floating-point number, and give the **reasons** for normalisation
- understand how **overflow** and **underflow** can occur
- understand that a binary representation is only an approximation, and that this gives rise to **rounding errors** 舍入误差

1 Worked examples

Study these first. Each one shows a **method** that a question later on this sheet needs.

Throughout, a number is stored as an 8-bit two's-complement **mantissa** with the binary point immediately after its sign bit, and an 8-bit two's-complement **exponent**.

■ Mantissa bits buy precision; exponent bits buy range



The total number of bits is fixed, so the two can only grow at each other's expense. That single fact answers the whole question:

- **more mantissa bits** – more significant figures, so greater **precision** and smaller rounding errors; but fewer exponent bits, so a **smaller range** of magnitudes.
- **more exponent bits** – a **larger range**, both larger and smaller magnitudes representable; but fewer mantissa bits, so **less precision**, and the values are more coarsely spaced.

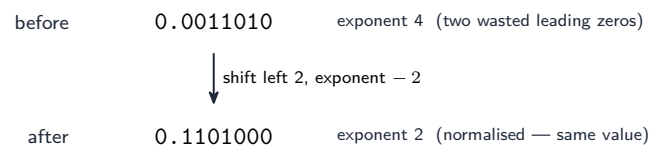
A three-mark answer names **both effects** and **both directions**. "More mantissa is more accurate" is one mark of three.

Worked example. *In a 10-bit mantissa, 6-bit exponent format, state the largest positive number.*

The largest mantissa is $0111111111 = 1 - 2^{-9}$, just under 1, and the largest exponent is $011111 = 31$. So the largest value is just under 2^{31}.

The smallest positive **normalised** number takes the smallest normalised mantissa, $0100000000 = 0.5$, with the most negative exponent, $100000 = -32$: $0.5 \times 2^{-32} = 2^{-33}$.

■ Normalisation, and why it exists



A **normalised** mantissa starts 01 for a positive number and 10 for a negative one: the first bit after the sign must be the **opposite** of the sign bit.

To normalise: shift the mantissa **left** until that holds, and **decrease** the exponent by one for every place shifted. The value is unchanged, because each left shift doubles the mantissa and each decrement of the exponent halves the result.

Worked example. *Normalise mantissa `00011000` with exponent `00000101`.*

The mantissa is +0.1875 and the exponent is 5, so the value is $0.1875 \times 32 = 6$.

Two left shifts give `01100000` = 0.75, and the exponent falls by 2 to 3. Check: $0.75 \times 8 = 6$. Same number.

The **reasons** for normalising, and a question asking for them wants the reason, not the rule:

- it gives the **greatest possible precision** for the bits available, because no mantissa bit is wasted holding a leading zero that carries no information;
- it makes the representation **unique**, so one value has exactly one stored form, which lets two numbers be compared directly.

■ Overflow and underflow both come from the exponent

- **Overflow**: the result of a calculation is **larger than the largest** value the format can hold; the exponent would need to be bigger than its bits allow.
- **Underflow**: the result is a non-zero value **smaller than the smallest** the format can hold, too close to zero for the exponent to express, so it is stored as **zero**.
- Both are limits of the **exponent's** range, not the mantissa's. Blaming the mantissa, or defining overflow as "too many digits", is the answer the mark scheme is looking out for.
- Widening the exponent postpones both, at the cost of precision – which is the trade-off above, seen from the other side.

■ Why a binary representation is only an approximation

A denary fraction is exact in binary only when its denominator is a power of 2. 0.5, 0.25 and 0.75 are exact; 0.1 is the recurring binary fraction 0.000110011…, which has to be cut off wherever the mantissa ends.

- **rounding errors** then accumulate over many operations, which is why `0.1 + 0.2` is not exactly `0.3`;
- so a real value is never tested with `=`. Test that the **difference** from the expected value is smaller than a small tolerance. A modulus function is **not** on the 9618 insert, so a question that needs one defines it; without one, test both directions, `IF X - 0.3 < 0.000001 AND 0.3 - X < 0.000001 THEN`;
- subtracting two nearly equal values leaves a result made almost entirely of error.
- Where exactness is required – currency, above all – use **fixed-point** 定点 or **BCD** instead.

2 Practice

Now use the methods above.

2.1 State what a **normalised** mantissa looks like for a positive number and for a negative number. [2]

2.2 A number is stored with mantissa `01100000` and exponent `00000010`.

Calculate its denary value. [2]

2.3 State **one** reason why floating-point numbers are normalised. [1]

2.4 State what is meant by **overflow** and by **underflow**. [2]

2.5 State which part of a floating-point number, the mantissa or the exponent, determines the **range** of values that can be stored. [1]

3 Exam-style questions

3.1 A system uses a fixed total of 16 bits for each floating-point number.

Describe the effect on the numbers that can be stored of **increasing** the number of bits allocated to the mantissa and reducing the number allocated to the exponent. [3]

3.2 A number is stored with mantissa `00011000` and exponent `00000101`.

(a) Calculate its denary value. [2]

(b) Write the number in **normalised** form, giving the new mantissa and the new exponent. [3]

(c) Show that your normalised form has the same value. [1]

3.3 Explain why **overflow** and **underflow** are caused by the **exponent** rather than by the mantissa. [3]

3.4 A program adds `0.1` to `0.2` and tests whether the result is equal to `0.3`. The test fails.

(a) Explain why. [2]

(b) State how the test should be written instead. [1]

3.5 A banking system stores currency amounts. Explain why floating-point representation is **not** suitable, and state what should be used instead. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- positive: the sign bit is `0` and the next bit is `1`, so it begins `01`
- negative: the sign bit is `1` and the next bit is `0`, so it begins `10`
- in both cases the bit after the sign is the **opposite** of the sign bit

2.2

- mantissa `01100000` $= 0.5 + 0.25 = 0.75$; exponent `00000010` $= 2$
- $0.75 \times 2^2 =$ **3**

2.3

- any one: it gives the greatest possible **precision** for the bits available, because no mantissa bit is wasted on a leading zero; or it makes the representation **unique**, so one value has one stored form

2.4

- **overflow**: the result is larger than the largest value the format can represent
- **underflow**: the result is a non-zero value smaller than the smallest the format can represent, so it is stored as zero

2.5

- the **exponent**

3.1

- **precision increases**: more mantissa bits means more significant figures, so values are stored more accurately and rounding errors are smaller
- **range decreases**: fewer exponent bits means the largest and smallest magnitudes that can be stored both move towards 1
- the total is fixed, so one can only be gained at the other's expense; overflow and underflow therefore occur for values that the old allocation could have held

3.2

(a) mantissa `00011000` $= 0.125 + 0.0625 = 0.1875$; exponent $= 5$

- $0.1875 \times 2^5 = 0.1875 \times 32 =$ **6**

(b) the mantissa must begin `01`, so shift **left** by **2** places

- new mantissa = **01100000**
- the exponent **decreases** by 2, from 5 to 3, so it is **00000011**
- shifting right, or increasing the exponent, changes the value and scores nothing

(c) $0.75 \times 2^3 = 0.75 \times 8 = 6$, the same value as in (a)

3.3

- the mantissa fixes the **precision** – how many significant figures are kept – and it always represents a value between -1 and $+1$
- the **exponent** fixes the **magnitude**, by saying what power of 2 the mantissa is multiplied by, so it alone decides how large or how small the number can be
- overflow happens when a result needs an exponent larger than the exponent field can hold, and underflow when it needs one more negative than the field can hold; running out of mantissa bits causes a loss of accuracy, not overflow

3.4

(a) 0.1 and 0.2 are **recurring** binary fractions, because their denominators are not powers of 2, so neither can be stored exactly

- each is rounded to fit the mantissa, and the two small errors survive into the sum, so the result differs from the stored value of `0.3` in its last bits

(b) test that the **difference** from `0.3` is smaller than a small tolerance rather than testing for zero, for example `IF X - 0.3 < 0.000001 AND 0.3 - X < 0.000001 THEN`

- a modulus function is not on the 9618 insert, so a question needing one supplies it

3.5

- currency values such as 0.10 are recurring binary fractions and cannot be stored exactly in floating-point
- the rounding errors accumulate over many transactions, so totals drift and accounts fail to balance – an error of any size is unacceptable in money
- use **fixed-point** representation, or **BCD**, in which each denary digit is stored exactly