Exercise walk-through

16.2.1

Recursive BNF, syntax diagrams, and RPN in both directions

A-Level Computer Science

You must be able to

- show understanding of how the grammar of a language can be expressed using **syntax diagrams** or **Backus-Naur Form** (BNF) notation, including **recursive** rules
- convert between a syntax diagram and a BNF rule
- show understanding of how RPN can be used to evaluate expressions, converting **infix to RPN** and **RPN back to infix**
- evaluate an RPN expression using a stack, showing the stack at each step

1 The method

Method — BNF, and the recursion that does the real work

A BNF rule defines one **non-terminal** in terms of terminals and other non-terminals:

```
<digit> ::= 0|1|2|3|4|5|6|7|8|9
```

That alone can only ever describe **one** character. To describe a number of **any length**, the rule must refer to **itself**:

```
<integer> ::= <digit> | <digit><integer>
```

Read it as: an integer is either a single digit, **or** a digit followed by another integer. That second alternative is the **recursion**, and it is what makes the rule unbounded.

- **Every recursive rule needs a non-recursive alternative** – here `<digit>` – or nothing can ever terminate. A rule written as `<integer> ::= <digit><integer>` alone defines nothing at all, and that is the error the mark scheme looks for.
- The recursive part goes on **one side**. `<digit><integer>` builds left to right; `<integer><digit>` builds right to left. Both describe the same set here, but mixing them in one alternative does not.

Method — BNF, and the recursion that does the real work

Worked example. *Write BNF for an identifier that starts with a letter and then contains any number of letters or digits.*

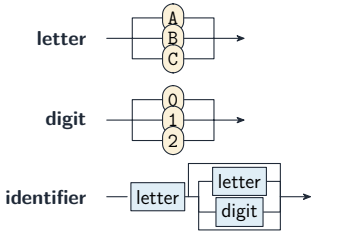
```
<letter>     ::=
↪  a|b|c|d|e|f|g|h|i|j|k|l|m|n|o|p|q|r|s|t|u|v|w|x|y|z
<digit>      ::= 0|1|2|3|4|5|6|7|8|9
<identifier> ::= <letter> | <identifier><letter> |
↪  <identifier><digit>
```

Method — BNF, and the recursion that does the real work

The base case is a single `<letter>`, which is what forces the first character to be a letter; the two recursive alternatives then add one character at a time to the right.

Method — BNF, and the recursion that does the real work

A BNF rule defines one non-terminal in terms of terminals and other non-terminals: [<digit> ::=...



the same rules in BNF

```
<letter> ::= A | B | C
<digit> ::= 0 | 1 | 2
<tail> ::= <letter> | <digit> | <letter><tail> | <digit><tail>
<identifier> ::= <letter> | <letter><tail>
```

a loop in the diagram becomes a rule that refers to itself (recursion); the first symbol is fixed by the first box on the rail
valid: `A`, `AB2`, `C01B` invalid: `2A` (starts with a digit), `AD` (D is not a letter here)

A syntax diagram and the BNF rule it corresponds to, side by side

Method — A syntax diagram is the same grammar, drawn

The translation between the two notations is mechanical:

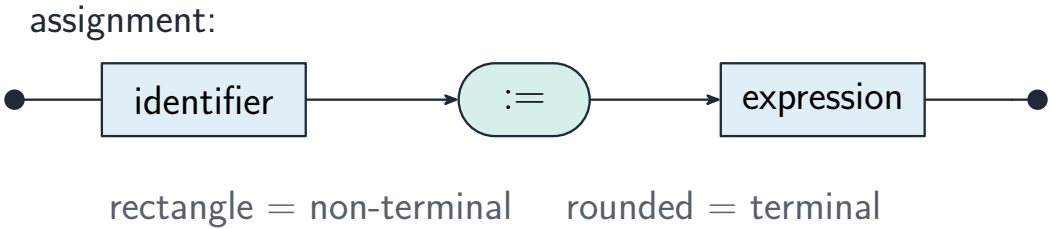
In BNF	**In a syntax diagram**
a **terminal**, such as 7	a **rounded** box (or circle)
a **non-terminal**, such as `<digit>`	a **rectangular** box
‘	‘, an alternative
one item after another	boxes in sequence along the line
recursion	a line that **loops back**

Method — A syntax diagram is the same grammar, drawn

So a loop in a diagram is a recursive rule in BNF, and a branch is a |. A question giving you one and asking for the other is asking you to apply that table.

Method — A syntax diagram is the same grammar, drawn

The translation between the two notations is mechanical: center So a loop in a diagram is a recursive rule in...



Method — RPN: why it exists, and both directions

In **Reverse Polish Notation** the operator comes **after** its two operands. It needs **no brackets** and no rules of precedence, because the order of the symbols already fixes the order of the operations – which is exactly why a compiler generates it.

Infix to RPN. Bracket the expression fully according to precedence, then move each operator to just after its own pair of operands.

The bracket is done first, so $3 + 4$ becomes `3 4 +`. That whole result is then multiplied by 2:

```
3 4 + 2 *
```

Convert $A + B \times C$. Multiplication binds tighter, so it is $A + (B \times C)$, giving `A B C * +` – **not** `A B + C *`, which is $(A + B) \times C$.

Method — RPN: why it exists, and both directions

RPN to infix. Work left to right; every time an operator appears, take the **two most recent** items, write them with the operator between them in brackets, and put that back.

Convert `4 5 + 2 6 - *.` → `(4 + 5)`, then `(2 - 6)`, then `((4 + 5) * (2 - 6))`.

Method — Evaluating with a stack

Read left to right. A **number is pushed**; an **operator pops two**, applies itself and pushes the result. The order matters: the **first** value popped is the **right-hand** operand.

Worked example. *Evaluate 5 2 − 3 ∗.*

Token	Action	Stack after
5	push 5	5
2	push 2	5, 2
−	pop 2 then 5; $5 - 2 = 3$; push	3
3	push 3	3, 3
∗	pop 3 then 3; $3 \times 3 = 9$; push	9

Method — Evaluating with a stack

The answer is the single value left on the stack: $\mathbf{9}$.

- For - and / the order is the whole question. Popping gives the **second** operand first, so it is second popped − first popped. Getting it backwards turns $5 - 2$ into $2 - 5$.

2 Practice

Practice 2.1 ▪ 2 marks

Write a BNF rule for <integer>, a whole number of one or more digits. You may use <digit> without defining it.

Solution 2.1

Method: BNF, and the recursion that does the real work — p.4

Worked steps

- `<integer> ::= <digit> | <digit><integer>` B1 B1
- one mark for the single-digit base case, one for the recursive alternative; `<integer><digit>` is equally acceptable

Practice 2.2 ▪ 1 mark

State why a recursive BNF rule must also have a **non-recursive** alternative.

Solution 2.2

Method: BNF, and the recursion that does the real work — p.4

Worked steps

- without a non-recursive alternative the rule refers to itself for ever and never reaches a terminal, so no string can be derived from it B1

Practice 2.3 ▪ 3 marks

In a syntax diagram, state what is shown by a **rounded** box, by a **rectangular** box, and by a line that **loops back**.

Solution 2.3

Method: A syntax diagram is the same grammar, drawn — p.8

Worked steps

- a **rounded** box is a **terminal** – a literal symbol that appears in the language itself B1
- a **rectangular** box is a **non-terminal** – a name defined by another rule B1
- a line that **loops back** is **repetition**, which in BNF is written as recursion B1

Practice 2.4 ▪ 2 marks

Convert the infix expression $A + B \times C$ to RPN.

Solution 2.4

Method: RPN: why it exists, and both directions — p.11

Worked steps

- multiplication binds more tightly, so the expression is $A + (B \times C)$ M1
- `A B C * +` A1
- `A B + C *` is $(A + B) \times C$ and is a different expression

Practice 2.5 ▪ 2 marks

Convert the RPN expression `7 2 3 * -` back to infix, and give its value.

Solution 2.5

Method: RPN: why it exists, and both directions — p.11

Worked steps

- the $*$ takes 2 and 3, and the $-$ then takes 7 and that result: $7 - (2 \times 3)$ B1
- $= \mathbf{1}$ B1

3 Exam-style

Exam-style 3.1 ▪ 4 marks

A programming language defines a variable name as starting with a letter, followed by any number of letters or digits.
Write the BNF rules for `<letter>`, `<digit>` and `<identifier>`.

Solution 3.1

> Method: BNF, and the recursion that does the real work — p.4

Worked steps

- `<letter> ::= a|b|c| ... |z` **B1**
- `<digit> ::= 0|1|2|3|4|5|6|7|8|9` **B1**
- `<identifier> ::= <letter>` as the base case, which forces the first character to be a letter **B1**
- `| <identifier><letter> | <identifier><digit>` as the recursive alternatives **B1**

Solution 3.1

```
<letter>     ::=
↪  a|b|c|d|e|f|g|h|i|j|k|l|m|n|o|p|q|r|s|t|u|v|w|x|y|z
<digit>      ::= 0|1|2|3|4|5|6|7|8|9
<identifier> ::= <letter> | <identifier><letter> |
↪  <identifier><digit>
```

- a rule whose base case is `<letter>|<digit>` allows an identifier to start with a digit, and loses the third mark

Exam-style 3.2 ▪ 3 marks

A student writes the rule `<integer> ::= <digit><integer>`.
Explain why this rule defines no valid integer, and correct it.

Solution 3.2

Method: BNF, and the recursion that does the real work — p.4

Worked steps

- the rule is **entirely recursive**: every alternative contains `<integer>` again B1
- so a derivation can never terminate – there is no way to reach a string made only of terminals B1
- add a non-recursive alternative: `<integer> ::= <digit> | <digit><integer>` B1

Exam-style 3.3 ▪ 3 marks

Convert the infix expression $(8 - 3) + (4/2)$ to Reverse Polish Notation.

Solution 3.3

Method: RPN: why it exists, and both directions — p.11

Worked steps

- each bracket is converted first: $8 - 3$ gives `8 3 -`, and $4/2$ gives `4 2 /` M1
- the + acts on those two results, so it comes last M1
- `8 3 - 4 2 / +` A1
- the value is $5 + 2 = 7$, which checks

Exam-style 3.4 ▪ 5 marks

Evaluate the RPN expression `4 5 + 2 6 - *` using a stack.
Show the contents of the stack after each token is processed.

Solution 3.4

Method: Evaluating with a stack — p.13

Worked steps

- 4 and 5 pushed, then + pops both and pushes 9 **B1**
- 2 and 6 pushed onto the 9 **B1**
- − pops 6 then 2, giving $2 - 6 = -4$, and pushes it **B1**
- $*$ pops -4 then 9, giving $9 \times (-4)$ **B1**
- the stack is left holding $\mathbf{-36}$ **B1**

Solution 3.4

Token	Stack after
4	4
5	4, 5
+	9
2	9, 2
6	9, 2, 6
-	9, -4
*	-36

Solution 3.4

- popping in the wrong order gives $6 - 2 = 4$ and a final answer of 36, with the sign lost

Exam-style 3.5 ▪ 2 marks

State **two** reasons why a compiler converts expressions into Reverse Polish Notation.

Solution 3.5

Method: RPN: why it exists, and both directions — p.11

Worked steps

any two:

- RPN needs **no brackets**, so the expression can be evaluated in a single left-to-right pass B1
- the **order of operations is fixed by the order of the symbols**, so no rules of precedence have to be applied while evaluating B1
- it is evaluated with a simple **stack**, which is straightforward to implement in machine code; and each operator is met only when both its operands are already available